Circuit Theory 1

Linear Circuits in Steady State | EE2021E

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Course Information

- One midterm exam
- One final exam
- Exam structure: 9 points for content and 1 point for presentation
- One A4 sheet of reference material is allowed in the exams
- Bonus points may be added to the coursework grade
  - Solving exercises and class discussions: **0 to 2 bonus points** (no points will be deducted for incorrect solutions or answers)
- Classroom rules
  - No attendance checks or attendance-based grading
  - Students may enter and leave freely and quietly (no permission required)
  - Do not make noise or cause distractions
  - For each reminder to maintain order: **0.25 points will be deducted from the entire class's coursework grade**
- **When requested, students are responsible for providing explanations and cooperating to clarify any concerns regarding their examination papers**

How to well study Circuit Theory

1. Do the exercises
2. Do the exercises
3. If you have not done the exercises, review the two points above

- Sources of exercises
  - In-class examples
  - Homework problems
  - Previous exams
  - Reference materials

References

1. Charles K. Alexander, Matthew N. O. Sadiku, *Fundamentals of Electric Circuits*, McGraw-Hill, 2021.
2. James Nilsson, Susan Riedel, *Electric Circuits*, 11^{th}, Pearson, 2020.
3. William H. Hayt, Jack E. Kemmerly, Steven M. Durbin, *Engineering Circuit Analysis*, 8^{th}, McGraw-Hill, 2012.
4. Mahmood Nahvi, Joseph A. Edminister, *Schaum's Outline of Electric Circuits*, 7^{th}, McGraw-Hill, 2018.

Structure of the Circuit Theory Course

Circuit Theory 1

1. Linear circuits in steady state

Circuit Theory 2

2. Linear circuits in transient state
3. Nonlinear circuits (steady state and transient state)
4. Transmission lines (steady state and transient state)

Course Outline

Chapter Outline

Chapter Outline

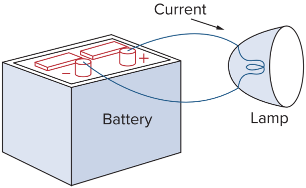
The Role of Circuit Theory

- General knowledge
  - Linear algebra
  - Single-variable calculus
  - Differential equations
  - Introductory physics (electricity)
- **Circuit Theory** ⇔ Signals and systems; automatic control theory
- Specialized knowledge
  - Analog electronics
  - Electrical machines
  - Power electronics
  - Electric drives
  - Power systems
  - etc.

The Modeling Approach

Problem

- Find the current flowing through the lamp



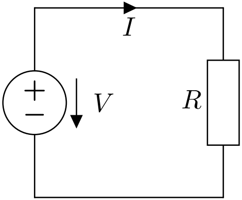
Approach $

- Buy battery, lamp, wires, and multimeter ⇒ build the circuit

Approach ☹

- From the physical and chemical properties of the battery ⇒ charges on the two electrodes ⇒ electric field in space ⇒ charges flowing in the wires and lamp ⇒ current

Approach ☺



$$I = \frac{V}{R}$$

⚠ Note: the *assumptions* for modeling

Chapter Outline

Basic Concepts and Quantities: Current

- Current is the rate of change of electric charge with respect to time
  Charge is in coulombs (C); current is in amperes (A)
  $$i = \frac{dq}{dt}$$
- Pay attention to the reference direction of current
  $$i_{ab}(t) = -i_{ba}(t)$$

Basic Concepts and Quantities: Voltage

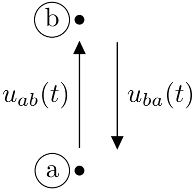
- The voltage (potential difference) between two points is the energy required to move one unit of charge from one point to the other

  Energy is measured in joules (J); voltage is measured in volts (V)

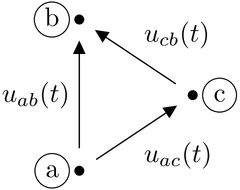
$$u_{ab} = \frac{dw_{ab}}{dq}$$

- Pay attention to the reference direction of the voltage arrow

$$u_{ab}(t) = -u_{ba}(t)$$



- Voltage “triangle” formula



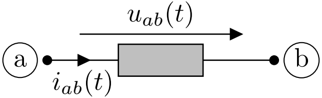
$$u_{ab}(t) = u_{ac}(t) + u_{cb}(t)$$

Some Basic Concepts and Quantities: Power

- Power is the rate of change of energy with respect to time
  Power is measured in watts (W)

$$p = \frac{dw}{dt} \Leftrightarrow p = \frac{dw}{dq} \cdot \frac{dq}{dt} = u \cdot i$$

$$\Rightarrow \text{ Power } = \text{ voltage } \times \text{ current}$$



- Instantaneous absorbed power: $p(t) = u_{ab}(t) \cdot i_{ab}(t)$
- Instantaneous supplied power: $-p(t) = u_{ab}(t) \cdot i_{ba}(t)$

⚠ Pay attention to the reference directions of current and voltage arrows and to the sign of power

- Average power over the time interval T:

$$P_{\text{avg}} = \frac{1}{T} \int_0^T p(t)dt$$

Chapter Outline

1. Concepts of Kirchhoff Circuit Model
   - Circuit Theory and the Modeling Approach
   - Basic Concepts and Quantities
   - Basic Circuit Elements
   - Circuit Models
   - Basic Problems in Electric Circuits
   - Basic Laws in Electric Circuits
   - Kirchhoff Equations System of a Circuit

Basic Elements: Characteristic Relations

- Circuit Theory is mainly concerned with two basic quantities: current and voltage; they are called the *state variables*

  (Power is derived from current and voltage)
- Each element is characterized by the relation between the current and the voltage on that element

$\Rightarrow$ This is called the *characteristic relation* of the element

- There are two forms of an element's characteristic relation
  1. $i(t) = f\left(u(t)\right)$ or $u(t) = f\left(i(t)\right)$
  2. $i(t) = f_i(t)$
     $u(t) = f_u(t)$

Basic Elements: Classification

Active and passive

- An *active element* can supply energy

  Examples: source elements and operational amplifiers (op-amps)
- A *passive element* can only absorb and/or store energy

  Examples: resistors, inductors, capacitors

Linear and nonlinear

- A *linear element* has a characteristic relation that is a linear function (algebraic or differential or integral)

  A circuit consisting only of linear elements is a linear circuit

  Recall: linearity means satisfying the *superposition principle*
- A *nonlinear element* has a nonlinear characteristic equation

  A circuit with at least one nonlinear element is a nonlinear circuit

Source Elements

A source is an element that supplies electromagnetic energy to a circuit

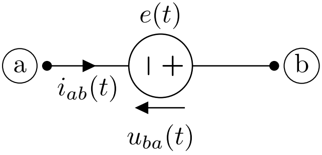
Types of electrical sources

- Voltage sources and current sources
- Independent sources and dependent sources

$\Rightarrow$ Four main combinations:
  - independent voltage source
  - independent current source
  - dependent voltage source
  - dependent current source
- Dependent sources are also called controlled sources
- Regarding control signal, dependent sources are classified:
  - voltage-controlled voltage source (VCVS)
  - current-controlled voltage source (CCVS)
  - voltage-controlled current source (VCCS)
  - current-controlled current source (CCCS)

Independent Voltage Source

- An independent voltage source supplies a voltage that does not depend on the current flowing through it
- Symbol:



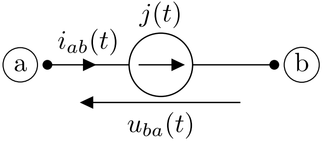
- Characteristic relation: $u_{ba}(t) = e(t)$, $\forall i_{ab}(t)$, $\forall t$

  Interpretation: the voltage of a voltage source is equal to the source electromotive force regardless of the current through it

⚠ Notes:
  - Other materials may use different symbols
  - This is an ideal voltage-source model; a practical voltage source can be modeled by an ideal voltage source and other elements
  - In practical circuits, never short-circuit a voltage source

Independent Current Source

- An independent current source supplies a current that does not depend on the voltage across it
- Symbol:



- Characteristic relation: $i_{ab}(t) = j(t)$, $\forall u_{ba}(t)$, $\forall t$

  Interpretation: the current of a current source is equal to the source current regardless of the voltage applied across it

⚠ Notes:
  - Other materials may use different symbols
  - This is an ideal current-source model; a practical current source can be modeled by an ideal current source and other elements
  - In practical circuits, never open-circuit a current source

Passive Elements

There are three types of passive elements

1. Energy-dissipating elements
   - Completely dissipate energy at all times
   - Do not supply or store energy
2. Electric-energy storage elements
   - Do not dissipate energy
   - Store energy in the form of electric charge
3. Magnetic-energy storage elements
   - Do not dissipate energy
   - Store energy in the form of magnetic flux

Resistor

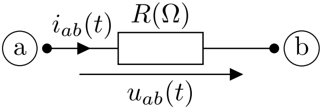
- A resistor is an energy-dissipating element; its unit is the ohm (Ω)

$\Rightarrow$ Absorbed power is always nonnegative: $p(t) = u_{ab}(t) \cdot i_{ab}(t) \geq 0$

$\Rightarrow$ There is a positive ratio between voltage and current on a resistor:

$$R = \frac{u_{ab}(t)}{i_{ab}(t)} \Leftrightarrow u_{ab}(t) = R \cdot i_{ab}(t),\ \forall i_{ab}(t),\ \forall t$$

- If R is constant $\Rightarrow$ linear resistor
- If $R = R(i)$ or $R = R(u)$ which is variable $\Rightarrow$ nonlinear resistor



Notes:

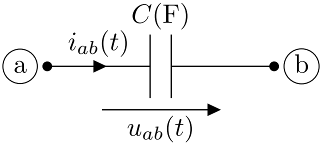
- Directions of current and voltage, and sign of absorbed power
- Conductance is the reciprocal of resistance; its unit is siemens (S)

$$G = \frac{1}{R} \Leftrightarrow i_{ab}(t) = G \cdot u_{ab}(t)$$

Capacitance (Capacitor)

- A capacitor stores energy in the form of electric charge $q_{ab}(t)$ proportional to the voltage $u_{ab}(t)$ through the coefficient C, called the capacitor's *capacitance*; its unit is the farad (F)

$$q_{ab}(t) = C(t) \cdot u_{ab}(t)$$



- From the relation between current and charge, the characteristic relation of a capacitor is:

$$i_{ab}(t) = \frac{dq_{ab}(t)}{dt} \Leftrightarrow i_{ab}(t) = \frac{d\left(C(t) \cdot u_{ab}(t)\right)}{dt}$$

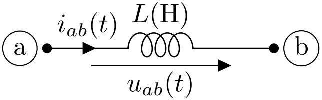
- If $C(t) = C$ is constant $\Rightarrow$ linear capacitor

$$i_{ab}(t) = C \cdot \frac{du_{ab}(t)}{dt}$$

Inductance (Inductor)

- An inductor stores energy in the form of magnetic flux $\Psi_{ab}(t)$ proportional to the current $i_{ab}(t)$ through the inductor's *inductance* L; its unit is the henry (H)

$$\Psi_{ab}(t) = L(t) \cdot i_{ab}(t)$$



- Variation of magnetic flux induces an electromotive force in the inductor $\Rightarrow$ the characteristic relation of an inductor is:

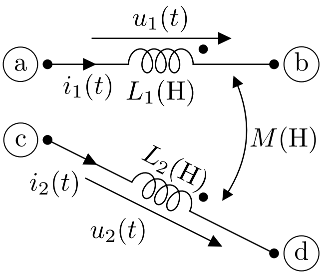
$$u_{ab}(t) = \frac{d\Psi_{ab}(t)}{dt} \Leftrightarrow u_{ab}(t) = \frac{d\left(L(t) \cdot i_{ab}(t)\right)}{dt}$$

- If $L(t) = L$ is constant $\Rightarrow$ linear inductor

$$u_{ab}(t) = L \cdot \frac{di_{ab}(t)}{dt}$$

Mutual Inductance Between Coils

- If there is flux in only one coil, the coefficient L is the coil's *self-inductance*
- If two or more coils have mutual flux linkages, there is a *mutual inductance* M between the coils



- Flux linkages:

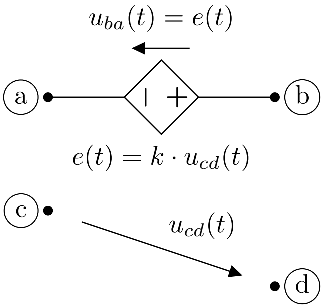
$$\begin{cases} \Psi_1(t) = L_1 \cdot i_1(t) \pm M \cdot i_2(t) \\ \Psi_2(t) = L_2 \cdot i_2(t) \pm M \cdot i_1(t) \end{cases}$$

- Linear self-inductance and mutual inductance:

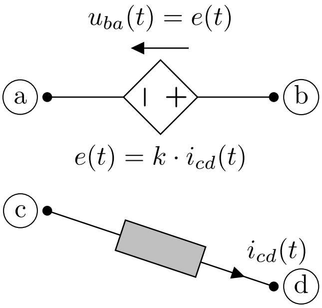
$$\begin{cases} u_1(t) = \dfrac{d\Psi_1(t)}{dt} = L_1 \cdot \dfrac{di_1(t)}{dt} \pm M \cdot \dfrac{di_2(t)}{dt} \\ u_2(t) = \dfrac{d\Psi_2(t)}{dt} = L_2 \cdot \dfrac{di_2(t)}{dt} \pm M \cdot \dfrac{di_1(t)}{dt} \end{cases}$$

Dependent Sources

Voltage-controlled voltage source

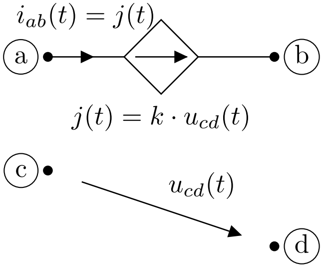


Current-controlled voltage source

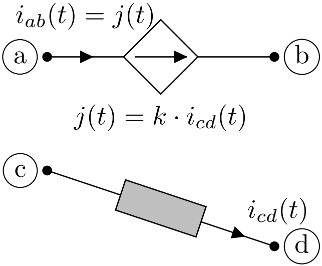


Dependent Sources

Voltage-controlled current source



Current-controlled current source

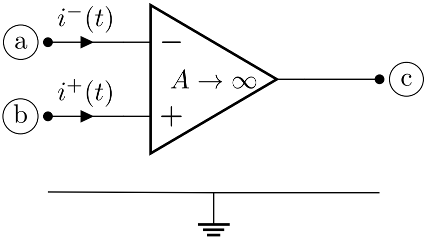


Operational Amplifier (Op-amp)

Operational amplifier

- A signal amplifier with a very large gain $A \sim 10^5 \to 10^7$
- Commonly used to implement operations such as $+ \quad - \quad \times \quad \div \quad d/dt \quad \int$
- Characteristic relation

$$\begin{cases} i^+ \approx 0;\ i^- \approx 0 \\ v_c(t) = A\left(v_b(t) - v_a(t)\right) \end{cases}$$



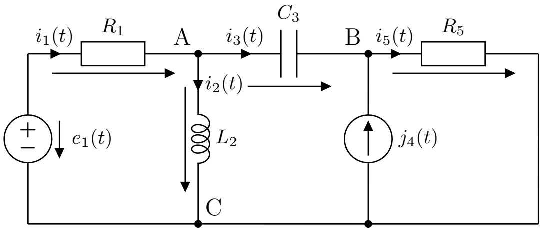
Ideal op-amp

- Gain $A = \infty$

$$\begin{cases} i^+ = 0;\ i^- = 0 \\ v_a(t) = v_b(t) \end{cases}$$

Chapter Outline

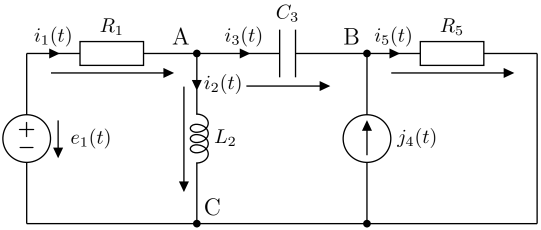
Circuit Models



Topology of a circuit

- A **node** is a connection point of two or more elements. The *degree* of a node is the number of elements connected to it. (If the terminal of an element is not connected to any other element, it is a 1st-order node.) In Circuit Theory, we usually focus on nodes of 3rd or higher order (called *essential nodes*).
- A **branch** is a path connecting two essential nodes
- A **loop** is a closed path through circuit branches

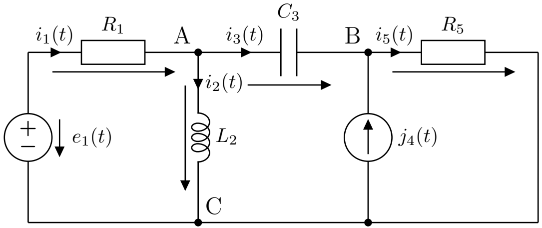
Circuit Models



Topology of a circuit

- A **tree** is a set of branches that connects all nodes of a circuit without forming any closed loop
- A **co-tree** consists of the remaining branches of the circuit that do not belong to the tree being considered
- A **fundamental loop** is the loop formed by one co-tree branch and branches of the tree being considered

Circuit Models



Some basic concepts

- **Branch current** is the current flowing through the elements of a branch
- **Node potential** is the voltage of that node relative to a node chosen as the reference point (the "ground" node); the ground potential is $v_{\text{gnd}} = 0$

Chapter Outline

1. Concepts of Kirchhoff Circuit Model
   - Circuit Theory and the Modeling Approach
   - Basic Concepts and Quantities
   - Basic Circuit Elements
   - Circuit Models
   - **Basic Problems in Electric Circuits**
   - Basic Laws in Electric Circuits
   - Kirchhoff Equations System of a Circuit

Basic Problems in Electric Circuits

Circuit analysis problem

- Given the circuit structure
- Given the values of the circuit elements
- Find the state-variable pair of current $i(t)$ and voltage $u(t)$ on the elements, then determine the power $p(t)$

Circuit synthesis (design) problem

- Given requirements for current $i(t)$, voltage $u(t)$, and power $p(t)$
- Propose a circuit structure
- Find the values of the circuit elements

Circuit Theory mainly considers circuit analysis problems.

Chapter Outline

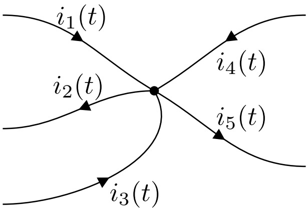
Basic Laws in Electric Circuits

Kirchhoff's current law (KCL)

- For any node: the algebraic sum of currents at the node is zero

  Convention: currents entering the node have sign +, currents leaving the node have sign −

  $\sum a_k \cdot i_k(t) = 0$; entering current: $a_k = 1$; leaving current: $a_k = -1$
- Equivalently: the sum of currents entering a node equals the sum of currents leaving that node: $\sum i_{\text{in}}(t) = \sum i_{\text{out}}(t)$

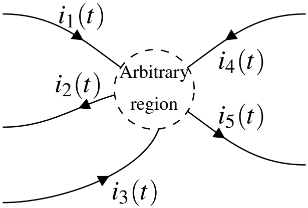


Basic Laws in Electric Circuits

Extended KCL

- For any closed circuit region: the sum of currents entering the region equals the sum of currents leaving it

$$\sum i_{\text{in}}(t) = \sum i_{\text{out}}(t)$$



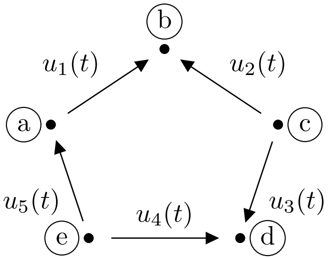
Basic Laws in Electric Circuits

Kirchhoff's voltage law (KVL)

- For any loop: the algebraic sum of voltages around the loop is zero
  Voltages in the reference direction of the loop have sign $+$
  Voltages opposite to the reference direction have sign $-$

$$\sum a_k \cdot u_k(t) = 0$$

same direction: $a_k = 1$; opposite direction: $a_k = -1$



Basic Laws in Electric Circuits

Some consequences of the two Kirchhoff laws

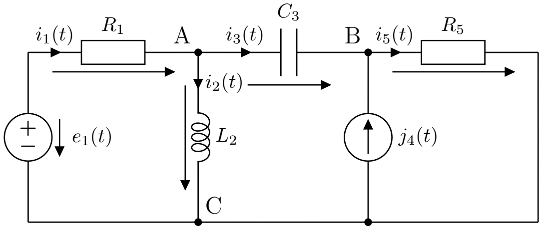
- Elements connected in series have the same current
- Elements connected in parallel have the same voltage
- Two different voltage sources must not be connected in parallel
- Two different current sources must not be connected in series

Basic Laws in Electric Circuits

Power conservation law

- At every instant, the total supplied power equals the total absorbed power
- For the circuit below

$$e_1(t)\cdot i_1(t)+j_4(t)\cdot u_5(t) = u_1(t)\cdot i_1(t)+u_2(t)\cdot i_2(t)+u_3(t)\cdot i_3(t)+u_5(t)\cdot i_5(t)$$



Basic Laws in Electric Circuits

Some other laws and properties

- Flux conservation in inductors
- Charge conservation in capacitors
- Superposition property of linear circuits

Chapter Outline

Kirchhoff Equations System of a Circuit

Variables and equations in a circuit analysis problem

- Unknown variables are currents and voltages on circuit elements
- For each element, if the current is known then the voltage can be determined, and vice versa (except source elements)
- On each branch, series-connected elements have the same current

$\Rightarrow$ If there are b unknown branches, set up b equations to find b variables

What type of equations?

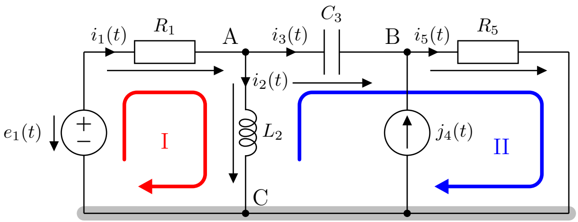
- A circuit has n essential nodes and l fundamental loops *that do not contain current sources*

$\Rightarrow$ $n-1$ equations from KCL

$\Rightarrow$ l equations from KVL

- It is easy to see that $l = b-(n-1) = b-n+1$

Kirchhoff Equations System of a Circuit



- Number of equations: $b = 4$
- Number of KCL equations: $3 - 1 = 2$
- Number of KVL equations: $4 - 2 = 2$

- Choose C as ground; write the KCL equations for nodes A and B

$$i_1(t) - i_2(t) - i_3(t) = 0$$
$$i_3(t) + j_4(t) - i_5(t) = 0$$

- Choose loops I and II as shown; write the KVL equations

$$u_1(t) + u_2(t) - e_1(t) = 0$$
$$u_3(t) + u_5(t) - u_2(t) = 0$$

Course Outline

Chapter Outline

2. Linear Circuits in Steady State with DC Sources
   - Kirchhoff Equation System of a DC Circuit
   - Branch-Current Method
   - Power in DC Circuits

Chapter Outline

Quantities in DC Circuits

- A DC circuit is a circuit whose sources are *DC sources*
- A DC source supplies a voltage or current that is *constant*

  (More generally, a DC source is a source whose current or voltage *does not change sign*. In this section, only the special constant case is considered.)

⇒ In *steady state* in a DC circuit, all current and voltage quantities are constant

$$u(t) = U = \text{const}$$
$$i(t) = I = \text{const}$$

⇒ Power in a DC circuit is also constant

$$p(t) = u(t) \cdot i(t) = U \cdot I = P = \text{const}$$

Elements in DC Circuits

- In DC circuits, the elements L and C degenerate

- Consider the characteristic relation of an inductor:

$u_L(t) = L \cdot \dfrac{di_L(t)}{dt}$; in a DC circuit $i_L(t) = I_L = \text{const}$

$\Rightarrow$ $u_L(t) = 0 \Leftrightarrow$ *the inductor is short-circuited*

- Consider the characteristic relation of a capacitor:

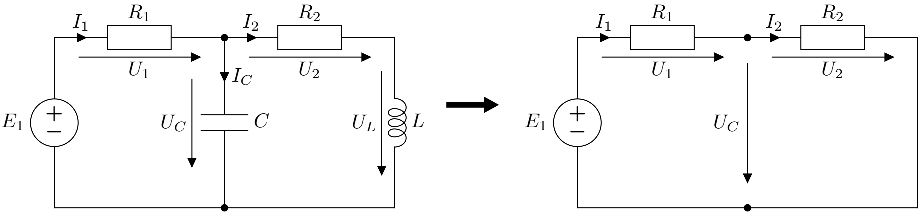
$i_C(t) = C \cdot \dfrac{du_C(t)}{dt}$; in a DC circuit $u_C(t) = U_C = \text{const}$

$\Rightarrow$ $i_C(t) = 0 \Leftrightarrow$ *the capacitor is open-circuited*

$\Rightarrow$ A DC circuit is equivalent to a **purely resistive** circuit

Kirchhoff Equation System of a DC Circuit

Example 2.1



$$i_L(t) = I_L = I_2 = I_1$$

$$u_C(t) = U_C = E_1 - U_1$$

Chapter Outline

Branch-Current Method

- The KCL and KVL equation systems contain both types of state variables: currents and voltages
- A circuit with b branches has $2b$ unknowns (b currents and b voltages) $\Rightarrow$ direct solution is inconvenient

$\Rightarrow$ We need to reduce the number of unknowns and equations

Branch-current method

- The unknown variables are the currents flowing in the branches

$\Rightarrow$ If the circuit has b unknown branches, set up b equations to find b branch currents

- Then use Ohm's law to determine voltages on branches and elements

Branch-Current Method

Procedure: Start from the Kirchhoff equation system

- The KCL equations already use branch currents as variables ⇒ keep them unchanged
- Convert the voltages in the KVL equations into relations of branch currents using Ohm's law

⚠ If there are dependent sources ⇒ must continue math operations to have an equation system with only branch-current variables.

⇒ Solve b equations in b branch-current unknowns ⇒ use Ohm's law to determine voltages on branches and elements

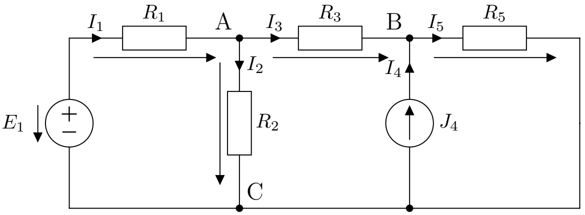
Summary

- Keep KCL in the form $\sum I$
- Convert KVL from the form $\sum U$ to the form $\sum (R \cdot I)$

⇒ Solve for I ⇒ determine U

Branch-Current Method

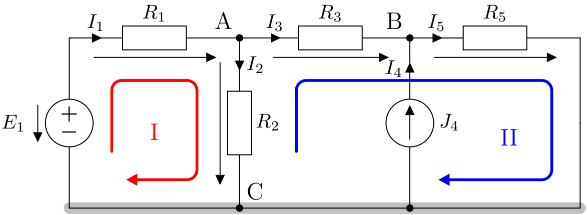
Example 2.2



- Number of KCL equations:
- Number of KVL equations:
- Choose nodes and current directions; write KCL equations:
- Choose loops and voltage directions; write KVL equations:

Branch-Current Method

Example 2.2



Kirchhoff equation system

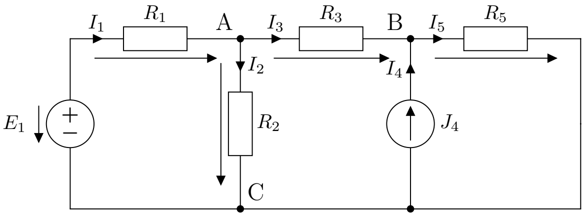
$$I_1 - I_2 - I_3 = 0$$
$$I_3 + J_4 - I_5 = 0$$
$$U_1 + U_2 - E_1 = 0$$
$$-U_2 + U_3 + U_5 = 0$$

Branch-current equation system

$$I_1 - I_2 - I_3 = 0$$
$$I_3 - I_5 = -J_4$$
$$R_1 \cdot I_1 + R_2 \cdot I_2 = E_1$$
$$-R_2 \cdot I_2 + R_3 \cdot I_3 + R_5 \cdot I_5 = 0$$

Branch-Current Method

Example 2.2



Substitute values:

- $E_1 = 15\,\text{V};\ R_1 = 5\,\Omega;$
- $R_2 = 8\,\Omega;\ R_3 = 6\,\Omega;$
- $J_4 = 1\,\text{A};\ R_5 = 10\,\Omega.$

Branch-current equation system

$$I_1 - I_2 - I_3 = 0$$
$$I_3 - I_5 = -1$$
$$5 \cdot I_1 + 8 \cdot I_2 = 15$$
$$-8 \cdot I_2 + 6 \cdot I_3 + 10 \cdot I_5 = 0$$

Branch-current results

$I_1 =$ A

$I_2 =$ A

$I_3 =$ A

$I_5 =$ A

Solution Method

Write in matrix form

$$\begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 5 & 8 & 0 & 0 \\ 0 & -8 & 6 & 10 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_5 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 15 \\ 0 \end{bmatrix}$$

Method 1: Use MATLAB (or equivalent tools such as Scilab)

```
>> A = [1 -1 -1 0; 0 0 1 -1; 5 8 0 0; 0 -8 6 10];
>> B = [0; -1; 15; 0];
>> X = inv(A)*B

X =

    1.1290
    1.1694
   -0.0403
    0.9597
```

- Should be used only to verify results
- Should not be overused, because it can weaken calculation skills

Solution Method

Method 2: Gaussian elimination (transform to upper triangular form)

$$\left|\begin{array}{cccc|c} 1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 \\ 5 & 8 & 0 & 0 & 15 \\ 0 & -8 & 6 & 10 & 0 \end{array}\right| \Leftrightarrow \left|\begin{array}{cccc|c} 1 & -1 & -1 & 0 & 0 \\ 0 & 13 & 5 & 0 & 15 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & -8 & 6 & 10 & 0 \end{array}\right| \Leftrightarrow$$

$$\left|\begin{array}{cccc|c} 1 & -1 & -1 & 0 & 0 \\ 0 & 13 & 5 & 0 & 15 \\ 0 & 0 & 9{,}0769 & 10 & 9{,}230 \\ 0 & 0 & 1 & -1 & -1 \end{array}\right| \Leftrightarrow \left|\begin{array}{cccc|c} 1 & -1 & -1 & 0 & 0 \\ 0 & 13 & 5 & 0 & 15 \\ 0 & 0 & 9{,}0769 & 10 & 9{,}230 \\ 0 & 0 & 0 & -2{,}101 & -2{,}0168 \end{array}\right|$$

$$\Rightarrow I_5 = -2{,}0168 \div (-2{,}101) = 0{,}9597\,\text{A}$$

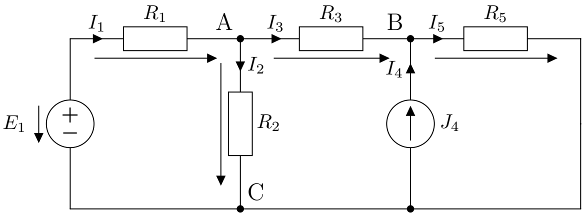
$$I_3 = (9{,}23 - 10 \times 0{,}9597) \div 9{,}0769 = -0{,}0404\,\text{A}$$

$$I_2 = (15 - 5 \times (-0{,}0404)) \div 13 = 1{,}1694\,\text{A}$$

$$I_1 = 1{,}1694 + (-0{,}0404) = 1{,}1290\,\text{A}$$

Branch-Current Method

Example 2.2



Substitute values:

- $E_1 = 15\,\text{V};\ R_1 = 5\,\Omega;$
- $R_2 = 8\,\Omega;\ R_3 = 6\,\Omega;$
- $J_4 = 1\,\text{A};\ R_5 = 10\,\Omega.$

Branch currents

$$I_1 = 1{,}1290\,\text{A}$$
$$I_2 = 1{,}1694\,\text{A}$$
$$I_3 = -0{,}0404\,\text{A}$$
$$I_5 = 0{,}9597\,\text{A}$$

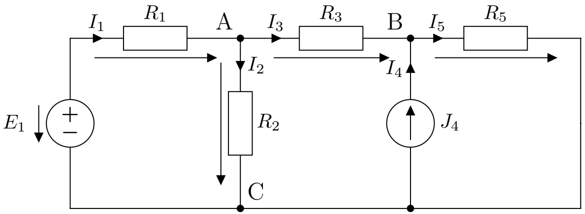
Voltages

$$U_1 = R_1 \cdot I_1 = 5 \times 1{,}1290 = 5{,}645\,\text{V}$$
$$U_2 = R_2 \cdot I_2 = 8 \times 1{,}1694 = 9{,}3552\,\text{V}$$
$$U_3 = R_3 \cdot I_3 = 6 \times -0{,}0404 = -0{,}2424\,\text{V}$$
$$U_5 = R_5 \cdot I_5 = 10 \times 0{,}9597 = 9{,}597\,\text{V}$$

Chapter Outline

Power in DC Circuits

Example 2.2



Substitute values:

- $E_1 = 15\,\text{V}$; $R_1 = 5\,\Omega$;
- $R_2 = 8\,\Omega$; $R_3 = 6\,\Omega$;
- $J_4 = 1\,\text{A}$; $R_5 = 10\,\Omega$.

Absorbed power on the resistors

- For a purely resistive circuit $P = U \cdot I = R \cdot I \cdot I = R \cdot I^2$

$\Rightarrow$ $P_{R_1} = 6{,}3732\,\text{W}$; $P_{R_2} = 10{,}94\,\text{W}$; $P_{R_3} = 0{,}00979\,\text{W}$; $P_{R_5} = 9{,}2102\,\text{W} \Rightarrow \sum P(\text{abs}) = 26{,}5332\,\text{W}$

Supplied power on the sources

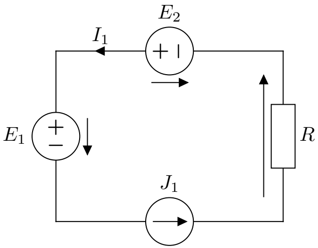
- $P_{E_1}(\text{sup}) = E_1 \cdot I_1 = 16{,}935\,\text{W}$; $P_{J_4}(\text{sup}) = J_4 \cdot U_5 = 9{,}597\,\text{W}$

Power in DC Circuits

- Dissipative elements (resistors) always absorb energy ($P_{\text{abs}} > 0$)
- Electric-storage elements (capacitors) and magnetic-storage elements (inductors) do not absorb energy; they only exchange energy through charging and discharging

$\Rightarrow$ If a circuit has only one source, that source must supply power
If there are multiple sources, some may supply power and others may absorb power

Example 2.3



- $E_1 = 15\,\text{V}$; $E_2 = 9\,\text{V}$; $R = 5\,\Omega$; $J_1 = 1\,\text{A}$

$\Rightarrow$ determine the power on the elements

Course Outline

Chapter Outline

Methods for Solving Linear Steady-State Circuits

Branch-current method

😄 Easy to understand and easy to set up equations

😵 Many unknowns and many equations ⇒ difficult to solve

⇒ Need to reduce the number of variables to simplify the problem

Approaches for simplifying the problem

- Introduce intermediate variables to simplify the equation system
  1. Loop Analysis
  2. Nodal Analysis
- Transform the circuit to simplify its structure
  3. Equivalent-resistance method
  4. Superposition method

Chapter Outline

3. Methods for Solving Linear Steady-State DC Circuits
   - Loop Analysis
   - Nodal Analysis
   - Equivalent-Resistance Method
   - Superposition Method

Loop Analysis

Variables and equations

- Instead of solving directly for branch currents, introduce "substituted unknowns" as dummy currents flowing in the *fundamental loops*

⇒ If the circuit has $l = b - n + 1$ fundamental loops *that do not contain current sources*, set up l equations to find l loop currents

- For each *current source*, a "discharging" loop is needed whose loop current equals the source current ⚠ Don't "discharge" through another current source

⇒ *Independent current source* doesn't increase the no. of equations

- Then determine each branch current as a combination of loop currents

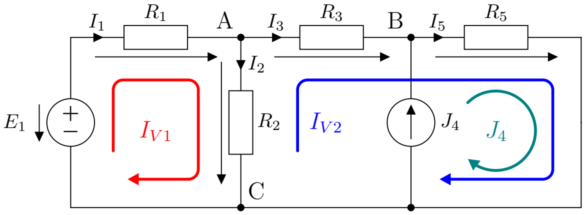
Procedure

- Set up l equations from KVL
- Assign a reference loop current to each fundamental loop
- Determine the "discharging" loop for current sources (if any)

Loop Analysis

Example 3.1

- The circuit has 4 branches and 3 nodes $\rightarrow$ it has $4 - 3 + 1 = 2$ fundamental loops
- Define 2 loop currents and 1 current-source discharging loop



Branch-current equations on loops from KVL

- Loop 1

  $R_1 \cdot I_1 + R_2 \cdot I_2 - E_1 = 0$
- Loop 2

  $-R_2 \cdot I_2 + R_3 \cdot I_3 + R_5 \cdot I_5 = 0$

Relation between branch currents and loop currents

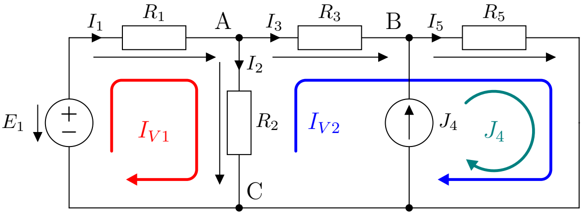
- $I_1 = I_{V1}$
- $I_2 = I_{V1} - I_{V2}$
- $I_3 = I_{V2}$
- $I_5 = I_{V2} + J_4$

Loop Analysis

Example 3.1

Substitute values

- $E_1 = 15\,\text{V}$; $R_1 = 5\,\Omega$;
- $R_2 = 8\,\Omega$; $R_3 = 6\,\Omega$;
- $J_4 = 1\,\text{A}$; $R_5 = 10\,\Omega$.



⇒ Loop-current equation system

$$(R_1 + R_2) \cdot I_{V1} - R_2 \cdot I_{V2} = E_1$$
$$-R_2 \cdot I_{V1} + (R_2 + R_3 + R_5) \cdot I_{V2} = -R_5 \cdot J_4$$

Substitute values → solve equations → loop currents → branch currents

$$\begin{aligned} 13 \cdot I_{V1} - 8 \cdot I_{V2} &= 15 \\ -8 \cdot I_{V1} + 24 \cdot I_{V2} &= -10 \end{aligned} \Rightarrow \begin{aligned} I_{V1} &= \quad \text{A} \\ I_{V2} &= \quad \text{A} \end{aligned} \Rightarrow \begin{aligned} I_1 &= \quad \text{A};\ I_2 = \quad \text{A}; \\ I_3 &= \quad \text{A};\ I_5 = \quad \text{A}. \end{aligned}$$

Solution Method

Matrix form of the equation system

$$\begin{bmatrix} 13 & -8 \\ -8 & 24 \end{bmatrix} \begin{bmatrix} I_{V1} \\ I_{V2} \end{bmatrix} = \begin{bmatrix} 15 \\ -10 \end{bmatrix}$$

- Method 1: Use MATLAB or Scilab (for cross-checking)
- Method 2: Elementary transformations or Gaussian elimination

Method 3: Use Cramer's rule

Equation system $A \cdot x = b$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Solution:

$$x_j = \frac{\det A_j}{\det A};$$

A_j: replace j^{th} column of matrix A by vector b

With $\det A = a_{11} \cdot a_{22} - a_{12} \cdot a_{21}$

Solution Method

Matrix form of the equation system

$$\begin{bmatrix} 13 & -8 \\ -8 & 24 \end{bmatrix} \begin{bmatrix} I_{V1} \\ I_{V2} \end{bmatrix} = \begin{bmatrix} 15 \\ -10 \end{bmatrix}$$

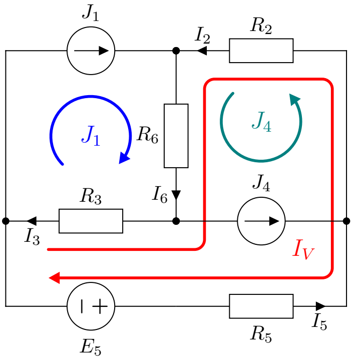
Apply Cramer's rule

$$I_{V1} = \frac{\begin{vmatrix} 15 & -8 \\ -10 & 24 \end{vmatrix}}{\begin{vmatrix} 13 & -8 \\ -8 & 24 \end{vmatrix}} = \frac{15 \cdot 24 - ((-8) \cdot (-10))}{13 \cdot 24 - ((-8) \cdot (-8))} = 1{,}129\,\mathrm{A}$$

$$I_{V2} = \frac{\begin{vmatrix} 13 & 15 \\ -8 & -10 \end{vmatrix}}{\begin{vmatrix} 13 & -8 \\ -8 & 24 \end{vmatrix}} = \frac{13 \cdot (-10) - (15 \cdot (-8))}{13 \cdot 24 - ((-8) \cdot (-8))} = -0.0403\,\mathrm{A}$$

Loop Analysis

Example 3.2



$J_1 = 1\,\text{A}$; $R_2 = 5\,\Omega$; $R_3 = 6\,\Omega$; $J_4 = 1{,}5\,\text{A}$; $R_5 = 4\,\Omega$; $E_5 = 12\,\text{V}$; $R_6 = 8\,\Omega$.

- Number of unknown branches: 4
- Number of nodes: 4

$\Rightarrow$ Number of fundamental loops: $4 - 4 + 1 = 1$

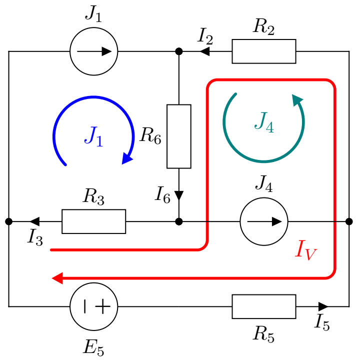
- Number of independent current sources: 2

$\Rightarrow$ Number of discharging loops: 2

- Branch-current equation on the fundamental loop:
  $-R_2 \cdot I_2 - R_3 \cdot I_3 - R_5 \cdot I_5 - R_6 \cdot I_6 + E_5 = 0$
- Replace branch-current variables by loop-current variables:
  $R_2 \cdot (-I_V + J_4) + R_3 \cdot (-I_V + J_1) - R_5 \cdot I_V + R_6 \cdot (-I_V + J_1 + J_4) = E_5$

Loop Analysis

Example 3.2



$J_1 = 1\,\text{A}$; $R_2 = 5\,\Omega$; $R_3 = 6\,\Omega$; $J_4 = 1{,}5\,\text{A}$; $R_5 = 4\,\Omega$; $E_5 = 12\,\text{V}$; $R_6 = 8\,\Omega$

- Simplify the loop-current equation $(R_2 + R_3 + R_5 + R_6) \cdot I_V = (R_3 + R_6) \cdot J_1 + (R_2 + R_6) \cdot J_4 - E_5$
- Substitute values to solve for the loop-current unknown

  $$23 \cdot I_V = 21{,}5 \Rightarrow I_V = 0{,}9348\,\text{A}$$
- From there, determine the branch currents, voltages, and power

⚠ Some “quick calculation” shortcuts can be found in the references

⚠ However, such shortcuts are easy to confuse in unusual cases
$\Rightarrow$ regular method starting from Kirchhoff equations is safer.

Chapter Outline

3. Methods for Solving Linear Steady-State DC Circuits
   - Loop Analysis
   - Nodal Analysis
   - Equivalent-Resistance Method
   - Superposition Method

Nodal Analysis

Variables and equations

- The substituted variables are the node potentials of the circuit

$\Rightarrow$ If the circuit has n *essential* nodes, set up $n-1$ equations to find $n-1$ node potentials

- Then determine branch currents from the node potentials

Procedure

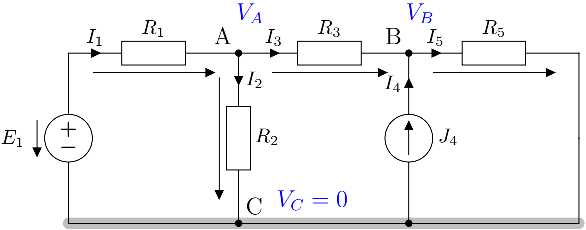
- Choose any node as the *ground* point with zero potential
- Set up $n-1$ equations from KCL

- This is the fundamental method for solving circuits containing op-amps

Nodal Analysis

Example 3.3

- The circuit has 3 nodes → it has $3 - 1 = 2$ KCL equations
- Choose point C as *ground*
- The unknowns are the potentials V_A and V_B



KCL equation system

- Node A: $I_1 - I_2 - I_3 = 0$
- Node B: $I_3 + J_4 - I_5 = 0$

⇒ Node-potential equation system

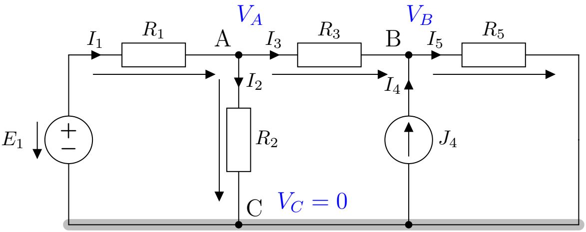
- $$\frac{-V_A + E_1}{R_1} - \frac{V_A}{R_2} - \frac{V_A - V_B}{R_3} = 0$$
- $$\frac{V_A - V_B}{R3} - \frac{V_B}{R5} = -J_4$$

Nodal Analysis

Example 3.3

Substitute values

- $E_1 = 15\,\text{V};\ R_1 = 5\,\Omega;$
- $R_2 = 8\,\Omega;\ R_3 = 6\,\Omega;$
- $J_4 = 1\,\text{A};\ R_5 = 10\,\Omega.$



Simplified node-potential equation system

$$\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)\cdot V_A - \frac{1}{R_3}\cdot V_B = \frac{E_1}{R_1}$$

$$\frac{1}{R_3}\cdot V_A - \left(\frac{1}{R_3} + \frac{1}{R_5}\right)\cdot V_B = -J_4$$

Substitute values

$$0{,}4917\cdot V_A - 0{,}1667\cdot V_B = 3$$

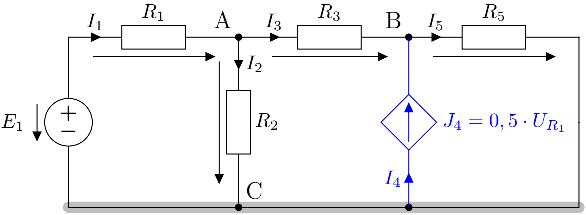
$$0{,}1667\cdot V_A - 0{,}2667\cdot V_B = -1$$

$$\Rightarrow \begin{matrix} V_A = 9{,}3548\,\text{V} \\ V_B = 9{,}5967\,\text{V} \end{matrix}$$

Nodal Analysis

Example 3.4: Case with a dependent source

- $E_1 = 15\,\mathrm{V}$; $R_1 = 5\,\Omega$;
- $R_2 = 8\,\Omega$; $R_3 = 6\,\Omega$;
- $R_5 = 10\,\Omega$.



$\Rightarrow$ Continue converting the dependent-source value into independent branch currents

KCL equation system

- Node A: $I_1 - I_2 - I_3 = 0$
- Node B: $I_3 + J_4 - I_5 = 0$

  $\Leftrightarrow I_3 + 2{,}5 \cdot I_1 - I_5 = 0$

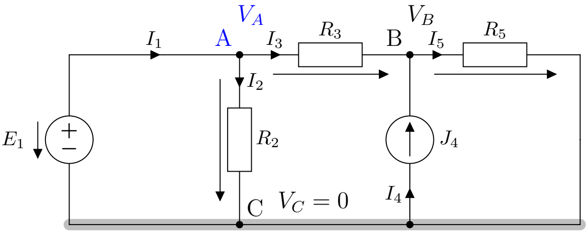
$\Rightarrow$ Node-potential equation system

- $\dfrac{-V_A + E_1}{R_1} - \dfrac{V_A}{R_2} - \dfrac{V_A - V_B}{R_3} = 0$
- $\dfrac{V_A - V_B}{R3} + 2{,}5 \cdot \dfrac{-V_A + E_1}{R_1} - \dfrac{V_B}{R5} = 0$

Nodal Analysis

Example 3.5: Supernode case

This is the case where circuit nodes are connected by a voltage source, with no “buffer” resistor

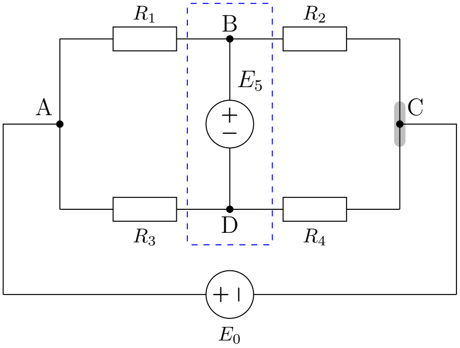


✗ Solve as in Example 3.3 and then set $R_1 = 0$
$\rightarrow$ indeterminate equation system $\rightarrow$ cannot be solved

✓ Potential $V_A = E_1$ $\rightarrow$ only one equation is needed to find potential V_B
$\rightarrow$ A supernode reduces the number of unknowns

Nodal Analysis

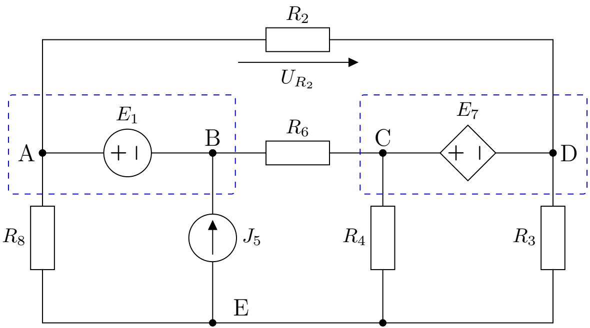
Example 3.6: Circuit with multiple supernodes



- Choose one node as the reference, e.g., node C $\Rightarrow V_A = E_0$
- Potential difference BD: $V_B - V_D = E_5$
- Apply the *extended KCL* to the closed region BD

Nodal Analysis

Example 3.7: Circuit with multiple supernodes and a dependent source



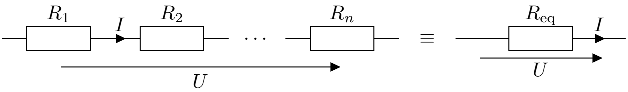
$E_1 = 20\,\text{V}$; $R_2 = 3\,\Omega$;
$R_3 = 1\,\Omega$; $R_4 = 4\,\Omega$;
$J_5 = 10\,\text{A}$; $R_6 = 6\,\Omega$;
$E_7 = 3 \cdot U_{R_2}$; $R_8 = 2\,\Omega$.

Chapter Outline

3. Methods for Solving Linear Steady-State DC Circuits
   - Loop Analysis
   - Nodal Analysis
   - Equivalent-Resistance Method
   - Superposition Method

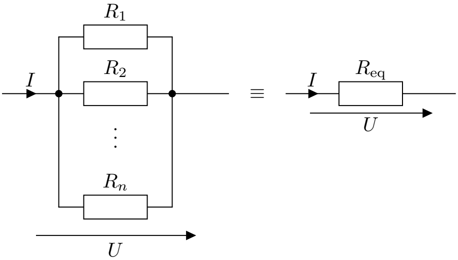
Equivalent-Resistance Method

Series circuit: equivalent resistance equals total resistance



$$R_{\text{eq}} = R_1 + R_2 + \ldots + R_n$$

Parallel circuit: equivalent conductance equals total conductance

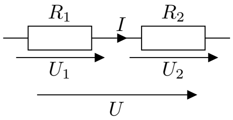


$$G_{\text{eq}} = G_1 + G_2 + \ldots + G_n$$

$$\Leftrightarrow \frac{1}{R_{\text{eq}}} = \frac{1}{R_1} + \frac{1}{R_2} + \ldots + \frac{1}{R_n}$$

Equivalent-Resistance Method

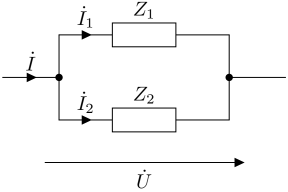
Voltage-division formula



- General formula: $U_k = \dfrac{R_k}{\sum_{i=1}^{n} R_i} \cdot U$
- Two-element case (common)

$$U_1 = \frac{R_1}{R_1 + R_2} \cdot U; \quad U_2 = \frac{R_2}{R_1 + R_2} \cdot U$$

Current-division formula

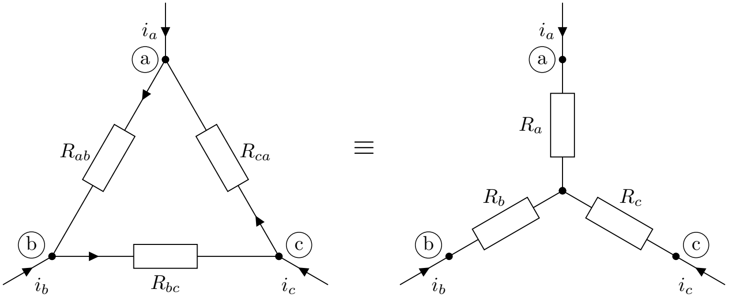


- General formula: $I_k = \dfrac{G_k}{\sum_{i=1}^{n} G_i} \cdot I$
- Two-element case (common)

$$I_1 = \frac{R_2}{R_1 + R_2} \cdot I; \quad I_2 = \frac{R_1}{R_1 + R_2} \cdot I$$

Equivalent-Resistance Method

Wye–delta transformation



$$\mathrm{Y} \to \Delta: \quad R_{ab} = R_a + R_b + \frac{R_a \cdot R_b}{R_c}, \quad R_{bc} = R_b + R_c + \frac{R_b \cdot R_c}{R_a}, \quad R_{ca} = R_c + R_a + \frac{R_c \cdot R_a}{R_b}$$

$$\Delta \to \mathrm{Y}: \quad R_a = \frac{R_{ca} \cdot R_{ab}}{R_{ab} + R_{bc} + R_{ca}}, \quad R_b = \frac{R_{ab} \cdot R_{bc}}{R_{ab} + R_{bc} + R_{ca}}, \quad R_c = \frac{R_{bc} \cdot R_{ca}}{R_{ab} + R_{bc} + R_{ca}}$$

Chapter Outline

3. Methods for Solving Linear Steady-State DC Circuits
   - Loop Analysis
   - Nodal Analysis
   - Equivalent-Resistance Method
   - Superposition Method

Superposition Method

- A linear system satisfies the *superposition principle*

$\Rightarrow$ The superposition principle can be used to simplify the analysis of linear circuits

Superposition in linear circuits with multiple sources

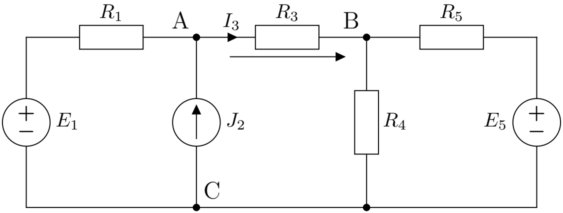
- Currents and voltages in the circuit are the *algebraic sum* of the currents and voltages produced by each *independent source* while the other *independent sources* are "turned off"

How to "turn off" (or "suppress") a source

- Independent voltage source $\rightarrow$ Short circuit
- Independent current source $\rightarrow$ Open circuit
- Do NOT turn off dependent sources

Superposition Method

Example 3.8: Determine current I_3 using superposition

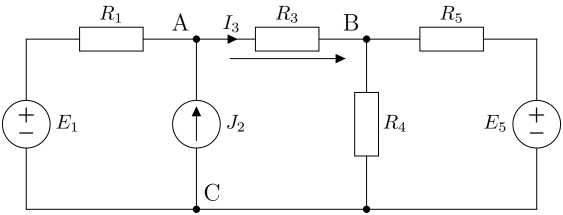


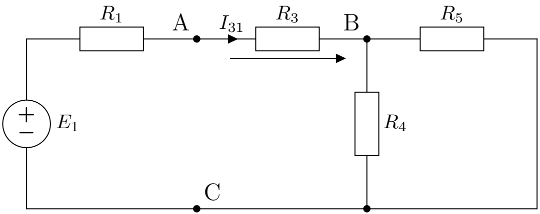
Solution

- $I_3 = I_{31} + I_{32} + I_{35}$
- I_{31} is the current through R_3 when only E_1 acts, with J_2 and E_5 off
- I_{32} is the current through R_3 when only J_2 acts, with E_1 and E_5 off
- I_{35} is the current through R_3 when only E_5 acts, with J_2 and E_1 off

Superposition Method

Example 3.8: Determine current I_3 using superposition





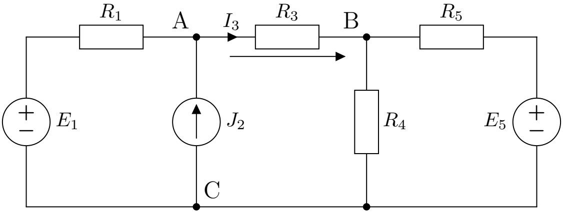
Component I_{31}

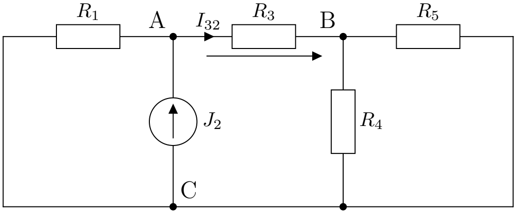
$$R_{\mathbf{eq}} = R_1 + R_3 + \frac{R_4 \cdot R_5}{R_4 + R_5}$$

$$I_{31} = \frac{E_1}{R_{\mathbf{eq}}}$$

Superposition Method

Example 3.8: Determine current I_3 using superposition





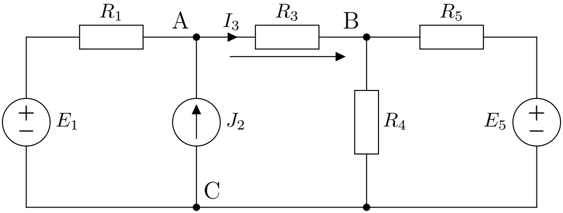
Component I_{32}

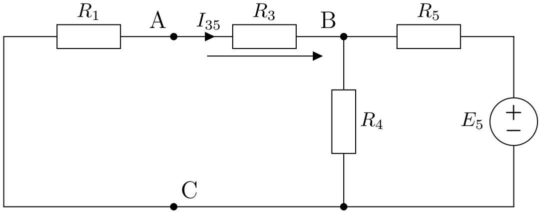
$$R_{345} = R_3 + \frac{R_4 \cdot R_5}{R_4 + R_5}$$

$$I_{32} = \frac{R_1}{R_1 + R_{345}} \cdot J_2$$

Superposition Method

Example 3.8: Determine current I_3 using superposition





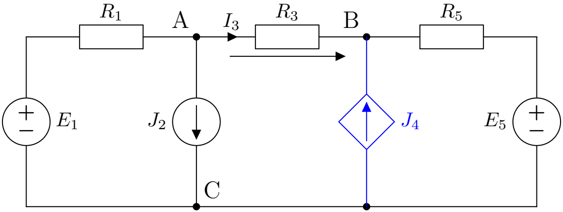
Component I_{35}

$$R_{\mathbf{eq}} = R_5 + \frac{R_4 \cdot (R_1 + R_3)}{R_1 + R_3 + R_4}$$

$$I_{32} = -\frac{E_5}{R_{\mathbf{eq}}} \cdot \frac{R_4}{R_1 + R_3 + R_4}$$

Superposition Method

Example 3.9: Superposition in a circuit with a dependent source



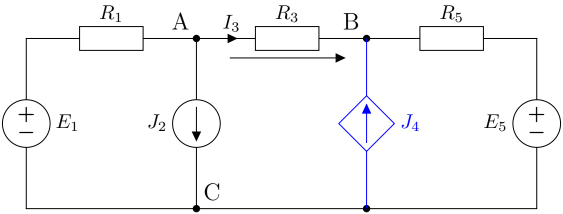
Solution

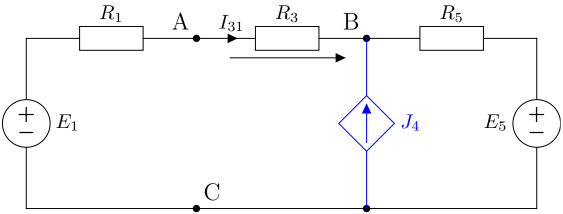
- $I_3 = I_{31} + I_{32}$
- I_{31} is the current through R_3 when only E_1 acts and J_2 off
- I_{32} is the current through R_3 when only J_2 acts and E_1 off

⚠ Do not turn off the dependent source J_4

Superposition Method

Example 3.9: Superposition in a circuit with a dependent source



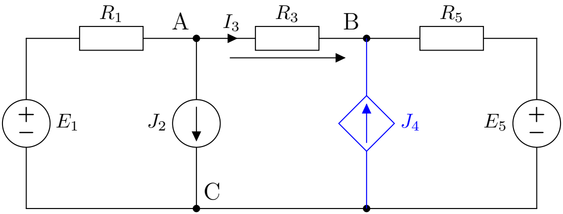


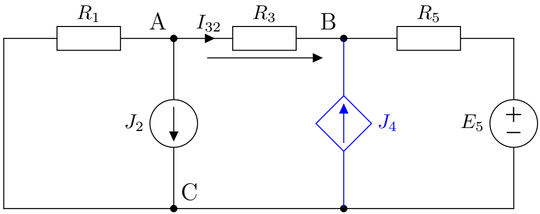
Component I_{31}

- Open-circuit source J_2
- Find I_{31} using a circuit-solving method (for example, nodal analysis)

Superposition Method

Example 3.9: Superposition in a circuit with a dependent source





Component I_{32}

- Short-circuit source E_1
- Find I_{32} using a circuit-solving method (for example, loop analysis)

Course Outline

Chapter Outline

4. Linear Circuits in Steady State with Sinusoidal Sources
   - Sinusoidal AC Quantities
   - Characteristic Equations of AC Circuit Elements
   - Kirchhoff Equations in AC Circuits
   - Using Complex Numbers to determine Sinusoidal Quantities
   - Phasor Representations of Circuit Elements
   - Phasor Representation of a Circuit and Kirchhoff Equation System

Chapter Outline

4. Linear Circuits in Steady State with Sinusoidal Sources
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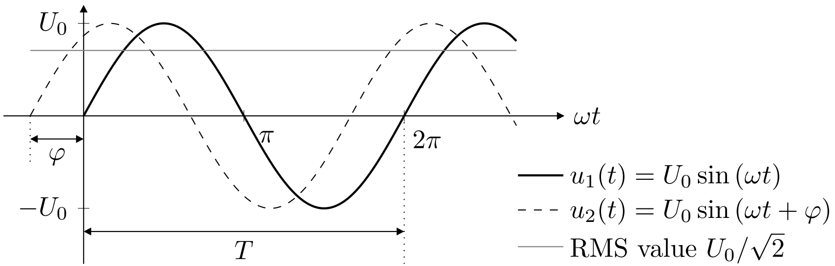
Sinusoidal AC Quantities

- AC quantity has instantaneous value whose sign changes over time
- If the change repeats with *period* T (*frequency* $f = 1/T$), it is called a *periodic* quantity
- If the periodic function is sin or cos, it is a *sinusoidal* function

$$u(t) = U_0 \sin(\omega t + \varphi)\,\mathrm{V}$$
$$i(t) = I_0 \sin(\omega t + \theta)\,\mathrm{A}$$

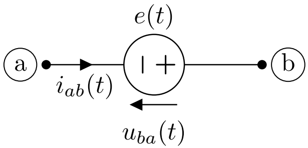
U_0, I_0: amplitude; φ, θ: phase angle;
ω: angular frequency;
$\omega = 2\pi f = 2\pi/T$



Chapter Outline

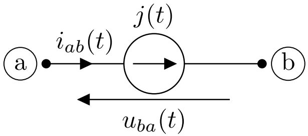
Characteristic Equation of AC circuit Element

Voltage source



- $u_{ba}(t) = e(t) = E_0 \sin(\omega t + \varphi), \forall i_{ab}(t), \forall t$

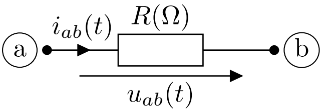
Current source



- $i_{ab}(t) = j(t) = J_0 \sin(\omega t + \theta), \forall u_{ba}(t), \forall t$

Characteristic Equation of AC circuit Element

Resistor



- $i_{ab}(t) = I_0 \sin(\omega t + \theta)$

$\Rightarrow$ $u_{ab}(t) = R \cdot i_{ab}(t) = R \cdot I_0 \sin(\omega t + \theta)$

Conductance

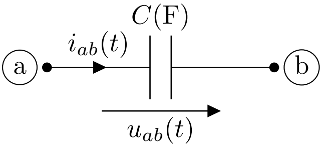
- $u_{ab}(t) = U_0 \sin(\omega t + \varphi)$

$\Rightarrow$ $i_{ab}(t) = G \cdot u_{ab}(t) = G \cdot U_0 \sin(\omega t + \varphi)$

$\Rightarrow$ Current and voltage on a resistor (and conductance) have the *same frequency* and are *in phase*

Characteristic Equation of AC circuit Element

Capacitance (capacitor)



- $u_{ab}(t) = U_0 \sin(\omega t + \varphi)$

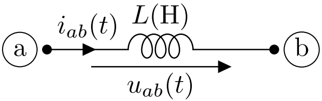
$$\Rightarrow i_{ab}(t) = C \cdot \frac{du_{ab}(t)}{dt} = C \cdot \frac{d}{dt}\left(U_0 \sin(\omega t + \varphi)\right)$$
$$= C \cdot U_0 \cdot \cos(\omega t + \varphi) \cdot \omega = \omega C \cdot U_0 \sin(\omega t + \varphi + 90^\circ)$$

$\Rightarrow$ Current and voltage on a capacitor have the *same frequency* and are *in quadrature*

- Capacitor current *leads* capacitor voltage by 90°
- Capacitor voltage *lags* capacitor current by 90°

Characteristic Equation of AC circuit Element

Inductance (inductor)



- $i_{ab}(t) = I_0 \sin(\omega t + \theta)$

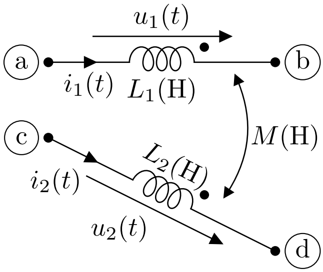
$$\Rightarrow u_{ab}(t) = L \cdot \frac{di_{ab}(t)}{dt} = L \cdot \frac{d}{dt}\left(I_0 \sin(\omega t + \theta)\right)$$
$$= L \cdot I_0 \cdot \cos(\omega t + \theta) \cdot \omega = \omega L \cdot I_0 \sin(\omega t + \theta + 90^\circ)$$

$\Rightarrow$ Current and voltage on an inductor have the *same frequency* and are *in quadrature*

- Inductor voltage *leads* current by 90°
- Inductor current *lags* voltage by 90°

Characteristic Equation of AC circuit Element

Mutual inductance between coils



- $i_1(t) = I_1 \sin(\omega t + \theta_1)$
- $i_2(t) = I_2 \sin(\omega t + \theta_2)$

$$\Rightarrow u_1(t) = L_1 \cdot \frac{di_1(t)}{dt} \pm M \cdot \frac{di_2(t)}{dt}$$

$$u_2(t) = L_2 \cdot \frac{di_2(t)}{dt} \pm M \cdot \frac{di_1(t)}{dt}$$

$$\Rightarrow u_1(t) = \omega L_1 \cdot I_1 \sin(\omega t + \theta_1 + 90^\circ) \pm \omega M \cdot I_2 \sin(\omega t + \theta_2 + 90^\circ)$$

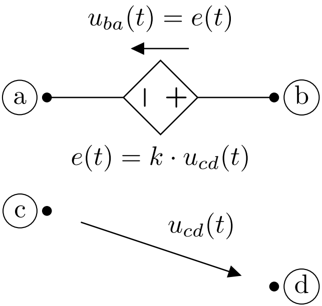
$$u_2(t) = \omega L_2 \cdot I_2 \sin(\omega t + \theta_2 + 90^\circ) \pm \omega M \cdot I_1 \sin(\omega t + \theta_1 + 90^\circ)$$

$\Rightarrow$ Currents and voltages on magnetically coupled coils have the *same frequency* and are *phase shifted* relative to one another

- The phase shift depends on the coupling sign and on θ_1 and θ_2

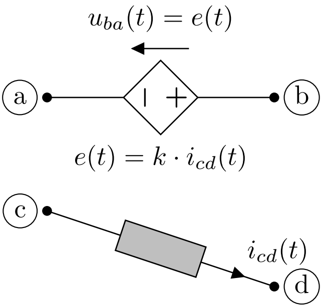
Characteristic Equation of AC circuit Element

Voltage-controlled voltage source



$\Rightarrow$ $e(t)$ is a sinusoidal function with the *same frequency* and *same phase* as $u_{cd}(t)$

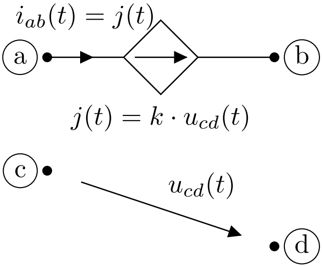
Current-controlled voltage source



$\Rightarrow$ $e(t)$ is a sinusoidal function with the *same frequency* and *same phase* as $i_{cd}(t)$

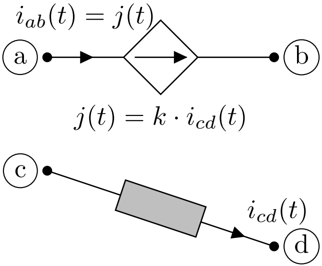
Characteristic Equation of AC circuit Element

Voltage-controlled current source



$\Rightarrow$ $j(t)$ is a sinusoidal function with the *same frequency* and *same phase* as $u_{cd}(t)$

Current-controlled current source



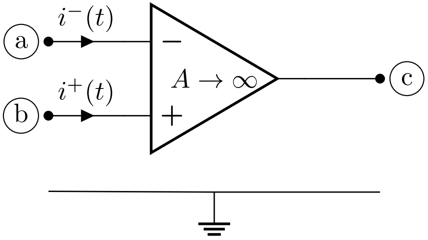
$\Rightarrow$ $j(t)$ is a sinusoidal function with the *same frequency* and *same phase* as $i_{cd}(t)$

Characteristic Equation of AC circuit Element

Operational amplifier (op-amp)

- All quantities in the circuit are sinusoidal functions with the *same frequency*
- The *phase angles* between quantities depend on the circuit structure and element values
- The characteristic relation keeps its general form

$$\begin{cases} i^+ \approx 0;\ i^- \approx 0 \\ v_c(t) = A\left(v_b(t) - v_a(t)\right) \end{cases}$$



Ideal op-amp

- Gain $A = \infty$

$$\begin{cases} i^+ = 0;\ i^- = 0 \\ v_a(t) = v_b(t) \end{cases}$$

Chapter Outline

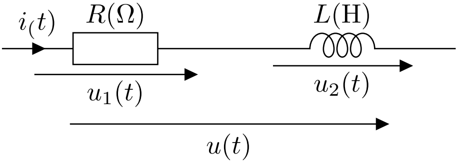
Kirchhoff Equations in AC circuits

- KCL and KVL are stated generally for all forms of current and voltage

⇒ KCL and KVL keep the forms $\sum i(t) = 0$ at a node and $\sum u(t) = 0$ around a loop, with $u(t)$ and $i(t)$ as sinusoidal functions

⚠ However, direct calculations with sinusoidal functions are not simple

Example 4.1: Determine voltage $u(t)$

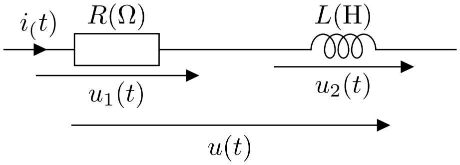


$$i(t) = 2\sin(5t + 20^\circ)$$

$$R = 6\,\Omega;\ L = 0{,}5\,\text{H}$$

Kirchhoff Equations in AC circuits

Example 4.1: Determine voltage $u(t)$



$i(t) = 2\sin(5t + 20°)$

$R = 6\,\Omega;\ L = 0{,}5\,\mathrm{H}$

- KVL equation: $u(t) = u_1(t) + u_2(t)$
- $u_1(t) = R \cdot i(t) = 12\sin(5t + 20°)$
- $u_2(t) = \omega L \cdot I_m \sin(\omega t + \varphi + 90°) = 5 \cdot 0{,}5 \cdot 2\sin(5t + 110°)$

$\Rightarrow$ $u(t) = 12\sin(5t + 20°) + 5\sin(5t + 110°)$

$$= \sqrt{12^2 + 5^2}\left(\frac{12}{\sqrt{12^2+5^2}}\sin(5t + 20°) + \frac{5}{\sqrt{12^2+5^2}}\cos(5t + 20°)\right)$$

$$= \sqrt{12^2 + 5^2}\sin(5t + 20° + 22{,}62°) = 13\sin(5t + 42{,}62°)\ \mathrm{V}$$

Chapter Outline

4. Linear Circuits in Steady State with Sinusoidal Sources
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Idea

- Calculations with trigonometric functions are difficult → convert to a form that is easier to determine

- Consider the AC quantity $f(t) = F_0 \sin(\omega t + \varphi)$
- If all current and voltage quantities are *sinusoidal functions*

$\Rightarrow$ Only 3 parameters, F_0, ω, and φ, are **sufficient to represent** $f(t)$

- If all sources in a linear circuit have the *same frequency* and considering that electrical engineering emphasizes the *RMS value*

$\Rightarrow$ Only 2 parameters, $F_0/\sqrt{2}$ and φ, are **sufficient to represent** $f(t)$

$\Rightarrow$ We need **a vector** with magnitude $F_0/\sqrt{2}$ and angle φ

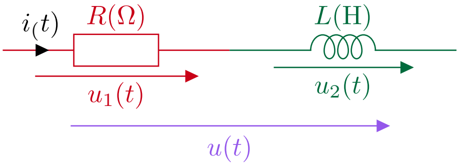
In the language of linear algebra, the *space* of same-frequency sinusoidal quantities in a circuit and the *space* of vectors of RMS values and phase angles are two *isomorphic* spaces.

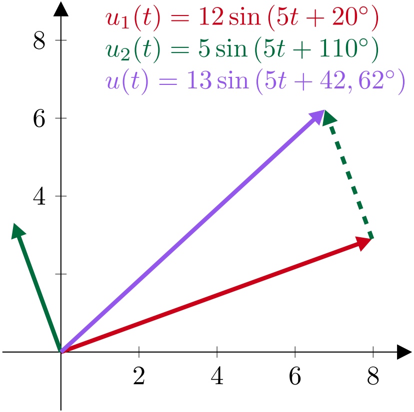
Phasor Concept

- Revisit Example 4.1

  $i(t) = 2\sin(5t + 20°)$

  $R = 6\,\Omega;\ L = 0{,}5\,\mathrm{H}$





- Geometric vector operations are still inconvenient for complex circuits

$\Rightarrow$ use *complex numbers* (Steinmetz, 1893)

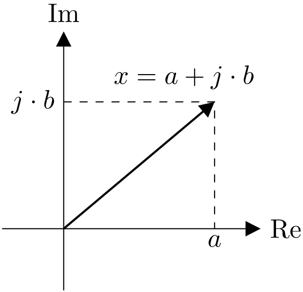
Review of Complex Numbers

- Complex numbers extend the set of real numbers by adding an *imaginary component*

$$x = a + j \cdot b$$

where (a,b) is a pair of real numbers and the imaginary operator is $j = \sqrt{-1}$

- A complex number has 2 components $\rightarrow$ it can be represented on the *complex plane* with the *real axis* Re and *imaginary axis* Im



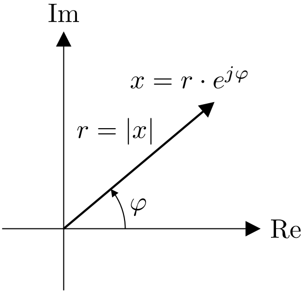
$$a = \mathrm{Re}(x)$$
$$b = \mathrm{Im}(x)$$

Review of Complex Numbers

- As a vector on the complex plane → it can be written in terms of *modulus* and *angle*

$$x = r \cdot e^{j\varphi}$$

- Modulus $r \geq 0$
- By convention, the positive direction and the value of φ are counterclockwise from the Re axis
- Conventional range of φ: $[-180°, 180°]$



$$r = |x| = \sqrt{a^2 + b^2}$$

$$\varphi = \arctan\left(\frac{b}{a}\right)$$

- Euler's formula

$$e^{j\varphi} = \cos\varphi + j \cdot \sin\varphi$$

Phasor Representation of a Sinusoidal Quantity

Phasor concept

- Let $\mathscr{P}$ be the **mapping** from the space of *same-frequency* sinusoidal functions in the **time domain** to the phasor space in the **complex domain**

$\Rightarrow$ The **phasor** of the quantity $f(t) = F_0 \sin(\omega t + \varphi)$ is

$$\dot{F} = \mathscr{P}(f(t)) = \frac{F_0}{\sqrt{2}} e^{j\varphi} = \frac{F_0}{\sqrt{2}} \underline{/\varphi}$$

⚠ Some phasor definitions use amplitude and/or the cosine function

Linearity of the phasor mapping

- Given a function $f(t) = a_1 \cdot f_1(t) + a_2 \cdot f_2(t) + \ldots + a_N \cdot f_N(t)$

$\Rightarrow$ $\dot{F} = \mathscr{P}(f(t)) = a_1 \cdot \mathscr{P}(f_1(t)) + a_2 \cdot \mathscr{P}(f_2(t)) + \ldots + a_N \cdot \mathscr{P}(f_N(t))$
$= a_1 \cdot \dot{F}_1 + a_2 \cdot \dot{F}_2 + \ldots + a_N \cdot \dot{F}_N$

Phasor Representation of a Sinusoidal Quantity

Phasor of differentiation

- Given $f(t) = F_0 \sin(\omega t + \varphi)$

$\Rightarrow$ $\dfrac{d}{dt} f(t) = \omega \cdot F_0 \sin(\omega t + \varphi + 90^\circ)$

$\Rightarrow$ Phasor of the derivative

$$\mathscr{P}\left(\frac{d}{dt} f(t)\right) = \mathscr{P}\left(\omega \cdot F_0 \sin(\omega t + \varphi + 90^\circ)\right) = \frac{\omega \cdot F_0}{\sqrt{2}} \angle \varphi + 90^\circ$$

$$= \frac{\omega \cdot F_0}{\sqrt{2}} e^{j(\varphi + 90^\circ)} = j\omega \frac{F_0}{\sqrt{2}} e^{j\varphi} = j\omega \frac{F_0}{\sqrt{2}} \angle \varphi$$

$$= j\omega \cdot \mathscr{P}(f(t))$$

$\Rightarrow$ The *phasor of the derivative* equals the phasor of the function *multiplied by* $j\omega$

Phasor Representation of a Sinusoidal Quantity

Phasor of integration

- Given $f(t) = F_0 \sin(\omega t + \varphi)$

$\Rightarrow \displaystyle\int f(t)dt = \frac{-1}{\omega} F_0 \sin(\omega t + \varphi + 90^\circ)$

$\Rightarrow$ Phasor of the integral

$$\mathscr{P}\left(\int f(t)dt\right) = \mathscr{P}\left(\frac{-1}{\omega} F_0 \sin(\omega t + \varphi + 90^\circ)\right) = \frac{-F_0}{\omega\sqrt{2}} \underline{/\varphi + 90^\circ}$$

$$= \frac{-F_0}{\omega\sqrt{2}} e^{j(\varphi+90^\circ)} = \frac{-j}{\omega} \cdot \frac{F_0}{\sqrt{2}} e^{j\varphi} = \frac{-j}{\omega} \cdot \frac{F_0}{\sqrt{2}} \underline{/\varphi}$$

$$= \frac{1}{j\omega} \cdot \mathscr{P}(f(t))$$

$\Rightarrow$ The *phasor of the integral* equals the phasor of the function *divided by $j\omega$*

Using Phasors to determine Sinusoidal Quantities

Example 4.2

- Given the currents $i_1(t) = 2\sin(5t + 20°)$ A; $i_2(t) = 3\sin(5t + 60°)$ A; $i_3(t) = 5\sin(5t - 15°)$ A determine $i_4(t) = i_1(t) + i_2(t) + i_3(t)$
- Phasor method

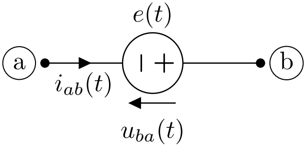
$$\dot{I}_4 = \dot{I}_1 + \dot{I}_2 + \dot{I}_3$$
$$= \frac{2}{\sqrt{2}}\angle 20° + \frac{3}{\sqrt{2}}\angle 60° + \frac{5}{\sqrt{2}}\angle -15°$$
$$= 5{,}972\angle 13{,}61° \text{ A}$$

$\Rightarrow$ $i_4(t) = 5{,}972\sqrt{2}\sin(5t + 13{,}61°)$ A

Chapter Outline

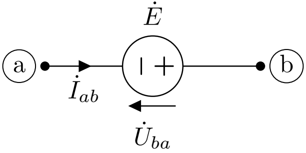
Independent Voltage Source

Independent voltage source in the time domain



$$u_{ba}(t) = e(t) = E_0 \sin\left(\omega t + \varphi\right),\ \forall i_{ab}(t),\ \forall t$$

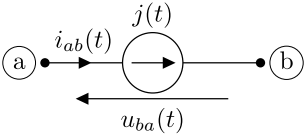
Phasor of an independent voltage source



$$\dot{U}_{ba} = \dot{E},\ \forall \dot{I}_{ab}$$

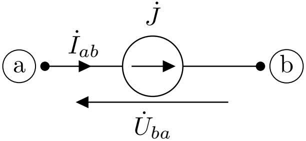
Independent Current Source

Independent current source in the time domain



$$i_{ab}(t) = j(t) = J_0 \sin(\omega t + \theta),\ \forall u_{ba}(t),\ \forall t$$

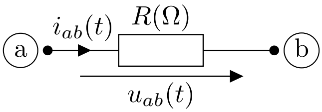
Phasor of an independent current source



$$\dot{I}_{ab} = \dot{J},\ \forall \dot{U}_{ba}$$

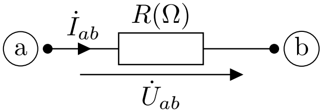
Resistor

Resistor (and conductance) in the time domain



$$u_{ab}(t) = R \cdot i_{ab}(t); \quad i_{ab}(t) = G \cdot u_{ab}(t)$$

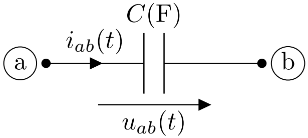
Phasor representation of a resistor (and conductance)



$$\dot{U}_{ab} = R \cdot \dot{I}_{ab}; \quad \dot{I}_{ab} = G \cdot \dot{U}_{ab};$$

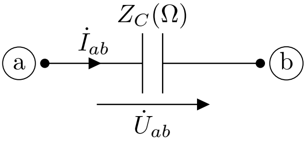
Capacitance (Capacitor)

Capacitor in the time domain



$$i_{ab}(t) = C \cdot \frac{du_{ab}(t)}{dt} = \omega C \cdot U_0 \sin\left(\omega t + \varphi + 90^\circ\right)$$

Phasor representation of a capacitor

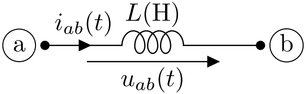


$$\dot{I}_{ab} = j\omega C \cdot \dot{U}_{ab}$$

$$\Leftrightarrow \dot{U}_{ab} = \frac{1}{j\omega C} \cdot \dot{I}_{ab} = Z_C \cdot \dot{I}_{ab}$$

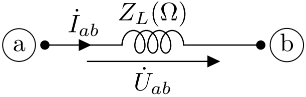
Inductance (Inductor)

Inductor in the time domain



$$u_{ab}(t) = L \cdot \frac{di_{ab}(t)}{dt} = \omega L \cdot I_0 \sin\left(\omega t + \theta + 90^\circ\right)$$

Phasor representation of an inductor

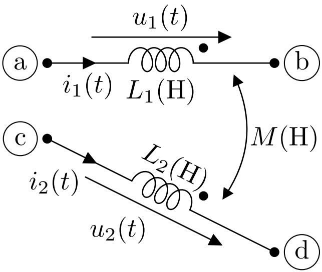


$$\begin{aligned}\dot{U}_{ab} &= j\omega L \cdot \dot{I}_{ab} \\ &= Z_L \cdot \dot{I}_{ab}\end{aligned}$$

$\Rightarrow$ The concept of **complex impedance** Z_C and Z_L

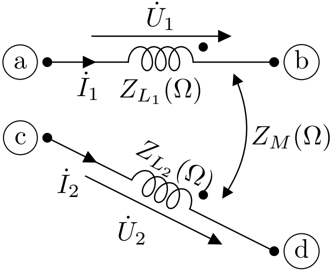
Mutual Inductance Between Two Coils

Mutual inductance in the time domain



$$u_1(t) = L_1 \cdot \frac{di_1(t)}{dt} \pm M \cdot \frac{di_2(t)}{dt}$$
$$u_2(t) = L_2 \cdot \frac{di_2(t)}{dt} \pm M \cdot \frac{di_1(t)}{dt}$$

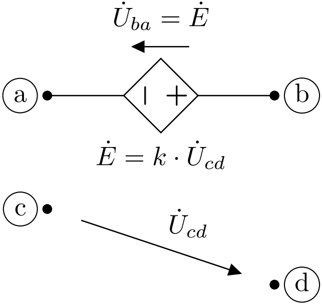
Phasor representation of mutual inductance



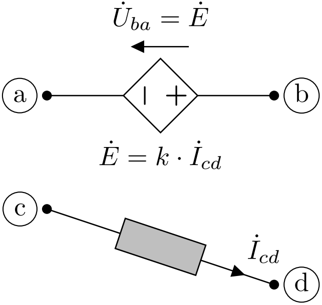
$$\begin{aligned}\dot{U}_1 &= j\omega L_1 \cdot \dot{I}_1 \pm j\omega M \cdot \dot{I}_2 \\ &= Z_{L_1} \cdot \dot{I}_1 \pm Z_M \cdot \dot{I}_2 \\ \dot{U}_2 &= j\omega L_2 \cdot \dot{I}_2 \pm j\omega M \cdot \dot{I}_1 \\ &= Z_{L_2} \cdot \dot{I}_2 \pm Z_M \cdot \dot{I}_1\end{aligned}$$

Phasor Representations of Dependent Sources

Voltage-controlled voltage source

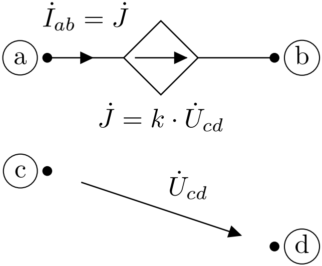


Current-controlled voltage source

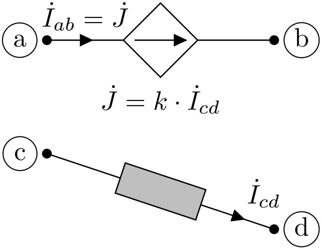


Phasor Representations of Dependent Sources

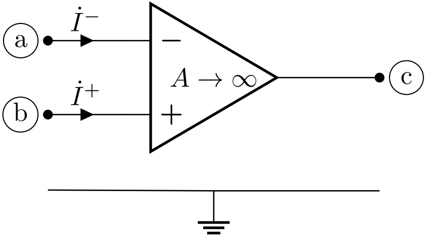
Voltage-controlled current source



Current-controlled current source



Phasor Representation of the Operational Amplifier



Practical op-amp

- The characteristic relation keeps its general form

$$\begin{cases} \dot I^+ \approx 0;\ \dot I^- \approx 0 \\ \dot V_c = A\left(\dot V_b - \dot V_a\right) \end{cases}$$

Ideal op-amp

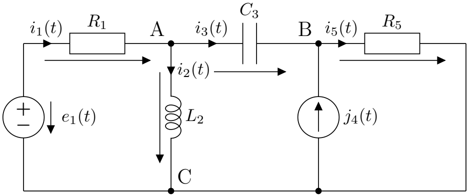
- Gain $A = \infty$

$$\begin{cases} \dot I^+ = 0;\ \dot I^- = 0 \\ \dot V_a = \dot V_b \end{cases}$$

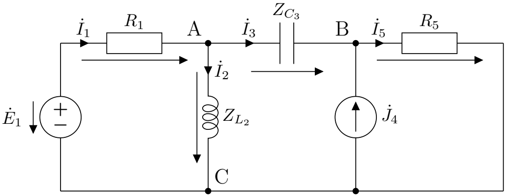
Chapter Outline

Phasor Representation of a Circuit

Circuit in the time domain

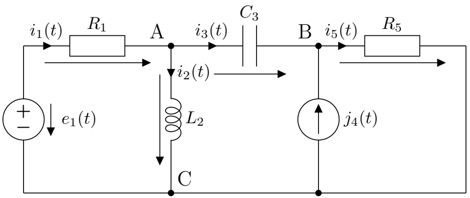


Phasor representation of the circuit



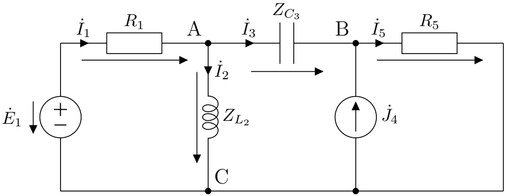
Phasor Representation of the Kirchhoff Equations

Circuit in the time domain



$$i_1(t) - i_2(t) - i_3(t) = 0$$
$$i_3(t) + j_4(t) - i_5(t) = 0$$
$$u_{R_1}(t) + u_{L_2}(t) - e_1(t) = 0$$
$$u_{C_3}(t) + u_{R_5}(t) - u_{L_2}(t) = 0$$

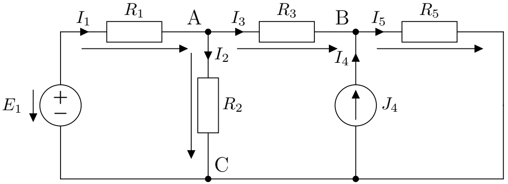
Phasor representation of the circuit



$$\dot{I}_1 - \dot{I}_2 - \dot{I}_3 = 0$$
$$\dot{I}_3 + \dot{J}_4 - \dot{I}_5 = 0$$
$$\dot{U}_{R_1} + \dot{U}_{L_2} - \dot{E}_1 = 0$$
$$\dot{U}_{C_3} + \dot{U}_{R_5} - \dot{U}_{L_2} = 0$$

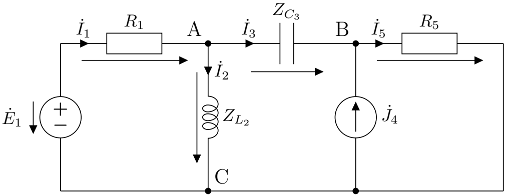
DC Circuits and Phasor Circuits

DC circuit



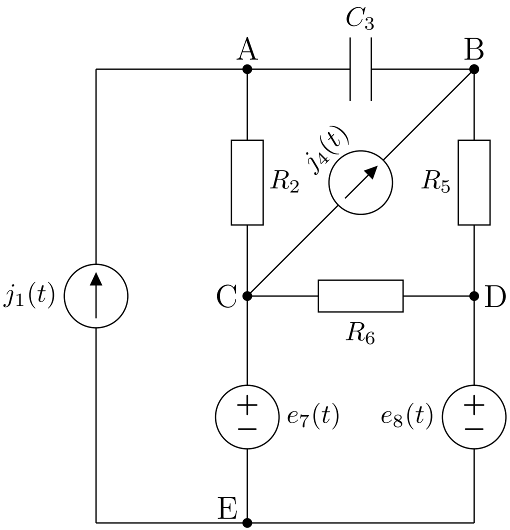
$$I_1 - I_2 - I_3 = 0$$
$$I_3 + J_4 - I_5 = 0$$
$$U_{R_1} + U_{R_2} - E_1 = 0$$
$$U_{R_3} + U_{R_5} - U_{R_2} = 0$$

Phasor representation of a AC circuit



$$\dot{I}_1 - \dot{I}_2 - \dot{I}_3 = 0$$
$$\dot{I}_3 + \dot{J}_4 - \dot{I}_5 = 0$$
$$\dot{U}_{R_1} + \dot{U}_{L_2} - \dot{E}_1 = 0$$
$$\dot{U}_{C_3} + \dot{U}_{R_5} - \dot{U}_{L_2} = 0$$

Exercises



$j_1(t) = 2\sin 50t$ A

$j_4(t) = 3\sin(50t + 30°)$ A

$e_7(t) = 80\sin(50t - 45°)$ V

$e_8(t) = 45\sin(50t + 90°)$ V

$R_2 = 20\,\Omega$; $R_5 = 16\,\Omega$

$R_6 = 24\,\Omega$; $C_3 = 0{,}4\,\text{mF}$

determine the impedances

Write the Kirchhoff equation systems

Course Outline

Chapter Outline

5. Methods for Solving Linear Steady-State AC circuits
   - Branch-Current Method
   - Loop Analysis
   - Nodal Analysis
   - Equivalent-Impedance Method
   - Superposition Method
   - Power in Sinusoidal AC Circuits

Solving Circuits in the Complex Domain

Characteristic equations in the time domain

$$u(t) = U_0 \sin(\omega t + \phi)$$
$$i(t) = I_0 \sin(\omega t + \theta)$$
$$u_L(t) = L\frac{di_L(t)}{dt}$$
$$i_C(t) = C\frac{du_C(t)}{dt}$$

$\xrightarrow{\text{Phasor}}$

Linear algebraic equations in the complex domain

$$\dot{U} = \frac{U_0}{\sqrt{2}}\angle\phi$$
$$\dot{I} = \frac{I_0}{\sqrt{2}}\angle\theta$$
$$\dot{U}_L = j\omega L\dot{I}_L = Z_L\dot{I}_L$$
$$\dot{U}_C = \frac{1}{j\omega C}\dot{I}_C = Z_C\dot{I}_C$$

$\downarrow + - \times \div$

Solution in the complex domain

$$\dot{U}\ ;\ \dot{I}$$

$\xleftarrow{\text{Inverse phasor}}$

Solution in the time domain

$$u(t)\ ;\ i(t)$$

(Time domain: $\downarrow + - \times \div$)

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5. Methods for Solving Linear Steady-State AC circuits
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   - Superposition Method
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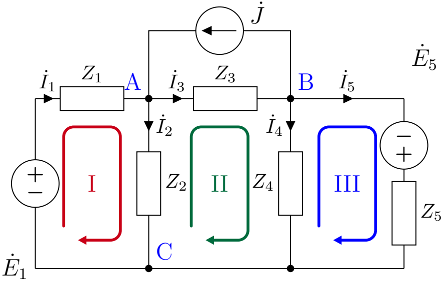
Branch-Current Method

Example 5.1

- 5 branches → 5 equations in 5 unknowns
- 3 nodes → set up $3-1=2$ KCL equations and $5-2=3$ KVL equations



Kirchhoff equation system

- Node A: $\dot{I}_1 - \dot{I}_2 - \dot{I}_3 + \dot{J} = 0$
- Node B: $\dot{I}_3 - \dot{I}_4 - \dot{I}_5 - \dot{J} = 0$
- Loop I: $\dot{U}_{Z_1} + \dot{U}_{Z_2} - \dot{E}_1 = 0$
- Loop II: $-\dot{U}_{Z_2} + \dot{U}_{Z_3} + \dot{U}_{Z_4} = 0$
- Loop III: $-\dot{U}_{Z_4} + \dot{U}_{Z_5} - \dot{E}_5 = 0$

Branch-current equation system

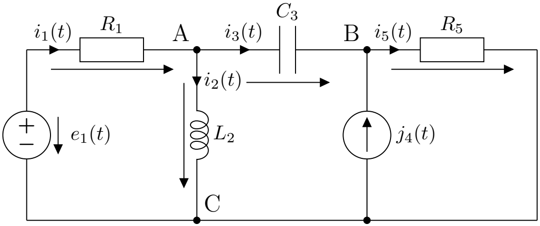
- A: $\dot{I}_1 - \dot{I}_2 - \dot{I}_3 + \dot{J} = 0$
- B: $\dot{I}_3 - \dot{I}_4 - \dot{I}_5 - \dot{J} = 0$
- I: $\dot{I}_1 Z_1 + \dot{I}_2 Z_2 - \dot{E}_1 = 0$
- II: $-\dot{I}_2 Z_2 + \dot{I}_3 Z_3 + \dot{I}_4 Z_4 = 0$
- III: $-\dot{I}_4 Z_4 - \dot{E}_5 + \dot{I}_5 Z_5 = 0$

Branch-Current Method

Example 5.2

Determine the currents using the branch-current method

$$e_1(t) = 12\sqrt{2}\sin(10t + 20^\circ)\,\mathrm{V};$$
$$R_1 = 5\,\Omega;\ L_2 = 0{,}3\,\mathrm{H};$$
$$C_3 = 0{,}02\,\mathrm{F};\ R_5 = 8\,\Omega;$$
$$j_4(t) = \sqrt{2}\sin(10t - 15^\circ)\,\mathrm{A}.$$



Branch-Current Method

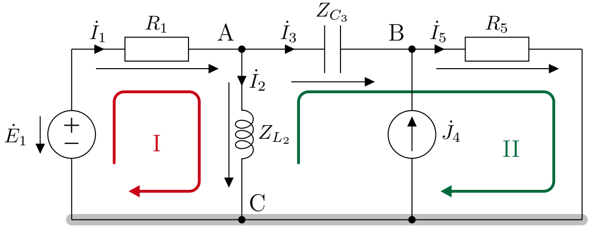
Example 5.2: convert to the complex domain

$\dot{E}_1 = 12\angle 20^\circ\,\text{V};$

$R_1 = 5\,\Omega;\ Z_{L_2} = j3\,\Omega;$

$Z_{C_3} = -j5\,\Omega;\ R_5 = 8\,\Omega;$

$\dot{J}_4 = 1\angle -15^\circ\,\text{A}.$



Kirchhoff equation system

- $\dot{I}_1 - \dot{I}_2 - \dot{I}_3 = 0$
- $\dot{I}_3 - \dot{I}_5 + \dot{J} = 0$
- $\dot{U}_{R_1} + \dot{U}_{L_2} - \dot{E}_1 = 0$
- $\dot{U}_{C_3} + \dot{U}_{R_5} - \dot{U}_{L_2} = 0$

Branch-current equation system

- $\dot{I}_1 - \dot{I}_2 - \dot{I}_3 = 0$
- $\dot{I}_3 - \dot{I}_5 + \dot{J} = 0$
- $R_1\dot{I}_1 + Z_{L_2}\dot{I}_2 - \dot{E}_1 = 0$
- $Z_{C_3}\dot{I}_3 + R_5\dot{I}_5 - Z_{L_2}\dot{I}_2 = 0$

Branch-Current Method

Matrix form of the equation system

- Substitute $\dot{I}_1 = \dot{I}_2 + \dot{I}_3$ and $\dot{I}_5 = \dot{I}_3 + \dot{J}$ into loops I and II

$$\begin{bmatrix} R_1 + Z_{L_2} & R_1 \\ -Z_{L_2} & R_5 + Z_{C_3} \end{bmatrix} \begin{bmatrix} \dot{I}_2 \\ \dot{I}_3 \end{bmatrix} = \begin{bmatrix} \dot{E}_1 \\ -R_5 \dot{J} \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} 5 + j3 & 5 \\ -j3 & 8 - j5 \end{bmatrix} \begin{bmatrix} \dot{I}_2 \\ \dot{I}_3 \end{bmatrix} = \begin{bmatrix} 12\underline{/20^\circ} \\ -8\underline{/-15^\circ} \end{bmatrix}$$

- Solve using Gaussian elimination or Cramer's rule

$$\dot{I}_1 = 1{,}6405\underline{/-22{,}25^\circ} \quad \Rightarrow i_1(t) = 1{,}6405\sqrt{2}\sin\left(10t - 22{,}25^\circ\right)\text{ A}$$

$$\dot{I}_2 = 2{,}6988\underline{/-27{,}07^\circ} \quad \Rightarrow i_2(t) = 2{,}6988\sqrt{2}\sin\left(10t - 27{,}07^\circ\right)\text{ A}$$

$$\dot{I}_3 = 1{,}0730\underline{/145{,}55^\circ} \quad \Rightarrow i_3(t) = 1{,}0730\sqrt{2}\sin\left(10t + 145{,}55^\circ\right)\text{ A}$$

$$\dot{I}_5 = 0{,}3575\underline{/76{,}88^\circ} \quad \Rightarrow i_5(t) = 0{,}3575\sqrt{2}\sin\left(10t + 76{,}88^\circ\right)\text{ A}$$

Chapter Outline

5. Methods for Solving Linear Steady-State AC circuits
   - Branch-Current Method
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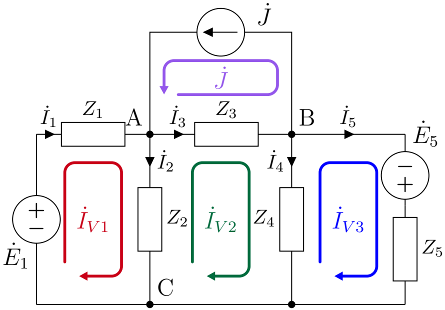
   - Nodal Analysis
   - Equivalent-Impedance Method
   - Superposition Method
   - Power in Sinusoidal AC Circuits

Loop Analysis



Example 5.3

- The circuit has 5 branches and 3 nodes $\rightarrow$ it has $5 - 3 + 1 = 3$ fundamental loops
- Define 3 loop currents and a current-source discharging loop

Branch-current equations on loops from KVL

- Loop 1: $\dot{I}_1 Z_1 + \dot{I}_2 Z_2 - \dot{E}_1 = 0$
- Loop 2: $-\dot{I}_2 Z_2 + \dot{I}_3 Z_3 + \dot{I}_4 Z_4 = 0$
- Loop 3: $-\dot{I}_4 Z_4 + \dot{I}_5 Z_5 - \dot{E}_5 = 0$

Relation between branch currents and loop currents

- $\dot{I}_1 = \dot{I}_{V1}$; $\dot{I}_2 = \dot{I}_{V1} - \dot{I}_{V2}$
- $\dot{I}_3 = \dot{I}_{V2} + \dot{J}$; $\dot{I}_4 = \dot{I}_{V2} - \dot{I}_{V3}$
- $\dot{I}_5 = \dot{I}_{V3}$

Loop Analysis

⇒ Matrix form of the loop-current equation system

$$\begin{bmatrix} Z_1+Z_2 & -Z_2 & 0 \\ -Z_2 & Z_2+Z_3+Z_4 & -Z_4 \\ 0 & -Z_4 & Z_4+Z_5 \end{bmatrix} \begin{bmatrix} \dot{I}_{V1} \\ \dot{I}_{V2} \\ \dot{I}_{V3} \end{bmatrix} = \begin{bmatrix} \dot{E}_1 \\ -Z_3\dot{J} \\ \dot{E}_5 \end{bmatrix}$$

Supplement: Sarrus' rule for computing a 3×3 determinant

- For matrix $A^{3\times 3}$

$$\left|\begin{array}{ccc|cc} a_{11} & a_{12} & a_{13} & \color{gray}{a_{11}} & \color{gray}{a_{12}} \\ a_{21} & a_{22} & a_{23} & \color{gray}{a_{21}} & \color{gray}{a_{22}} \\ a_{31} & a_{32} & a_{33} & \color{gray}{a_{31}} & \color{gray}{a_{32}} \end{array}\right. \qquad \begin{array}{c} - \\ \\ + \end{array}$$

⇒ The determinant of the matrix

$$\det A = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12}$$

⚠ This rule is NOT valid for determinants of order ≥ 4

Loop Analysis

Example 5.4

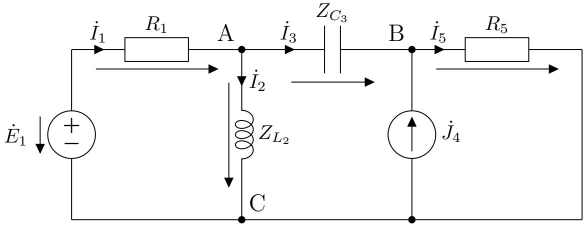
Solve the circuit using the loop analysis

$\dot{E}_1 = 12\underline{/20^\circ}\,\text{V};$

$R_1 = 5\,\Omega;\ Z_{L_2} = j3\,\Omega;$

$Z_{C_3} = -j5\,\Omega;\ R_5 = 8\,\Omega;$

$\dot{J}_4 = 1\underline{/-15^\circ}\,\text{A}.$



Branch-current equations on loops from KVL

- $R_1\dot{I}_1 + Z_{L2}\dot{I}_2 - \dot{E}_1 = 0$
- $Z_{C_3}\dot{I}_3 + R_5\dot{I}_5 - Z_{L_2}\dot{I}_2 = 0$

Relation between branch currents and loop currents

- $\dot{I}_1 = \dot{I}_{V1}$; $\dot{I}_2 = \dot{I}_{V1} - \dot{I}_{V2}$
- $\dot{I}_3 = \dot{I}_{V2}$; $\dot{I}_5 = \dot{I}_{V2} + \dot{J}$

Loop Analysis

Matrix form of the loop-current equation system

$$\begin{bmatrix} R_1 + Z_{L_2} & -Z_{L_2} \\ -Z_{L_2} & Z_{L_2} + Z_{C_3} + R_5 \end{bmatrix} \begin{bmatrix} \dot{I}_{V1} \\ \dot{I}_{V2} \end{bmatrix} = \begin{bmatrix} \dot{E}_1 \\ -R_5 \dot{J} \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} 5 + j3 & -j3 \\ -j3 & 8 - j2 \end{bmatrix} \begin{bmatrix} \dot{I}_{V1} \\ \dot{I}_{V2} \end{bmatrix} = \begin{bmatrix} 12\underline{/20^\circ} \\ -8\underline{/-15^\circ} \end{bmatrix}$$

- Solve the system using algebraic methods

  $\dot{I}_{V1} = 1{,}6403\underline{/-22{,}25^\circ}\,\mathrm{A}$

  $\dot{I}_{V2} = 1{,}0730\underline{/145{,}55^\circ}\,\mathrm{A}$

$\Rightarrow$ determine the branch currents and other quantities

Chapter Outline

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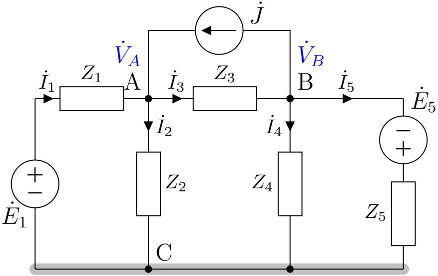
Nodal Analysis

Example 5.5

- The circuit has 3 nodes → $3 - 1 = 2$ KCL equations
- Choose point C as *ground*
- The unknowns are the potentials $\dot{V}_A$ and $\dot{V}_B$



KCL equation system

- Node A: $\dot{I}_1 - \dot{I}_2 - \dot{I}_3 + \dot{J} = 0$

$$\Leftrightarrow \frac{\dot{U}_{Z1}}{Z_1} - \frac{\dot{U}_{Z2}}{Z_2} - \frac{\dot{U}_{Z3}}{Z_3} = -\dot{J}$$

- Node B: $\dot{I}_3 - \dot{I}_4 - \dot{I}_5 - \dot{J} = 0$

$$\Leftrightarrow \frac{\dot{U}_{Z3}}{Z_3} - \frac{\dot{U}_{Z4}}{Z_4} - \frac{\dot{U}_{Z5}}{Z_5} = \dot{J}$$

Node-potential equation system

- $$\frac{-\dot{V}_A + \dot{E}_1}{Z_1} - \frac{\dot{V}_A}{Z_2} - \frac{\dot{V}_A - \dot{V}_B}{Z_3} = -\dot{J}$$
- $$\frac{\dot{V}_A - \dot{V}_B}{Z_3} - \frac{\dot{V}_B}{Z_4} - \frac{\dot{V}_B + \dot{E}_5}{Z_5} = \dot{J}$$

Nodal Analysis

Example 5.6

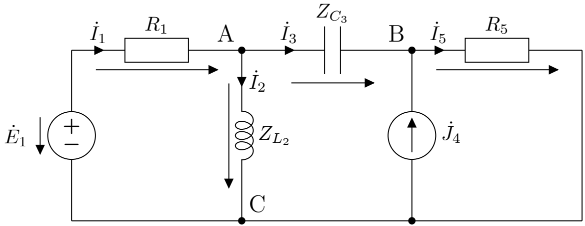
Solve the circuit below using the nodal analysis

$\dot{E}_1 = 12\angle 20^\circ$ V;

$R_1 = 5\,\Omega$; $Z_{L_2} = j3\,\Omega$;

$Z_{C_3} = -j5\,\Omega$; $R_5 = 8\,\Omega$;

$\dot{J}_4 = 1\angle -15^\circ$ A.



KCL equation system

- $\dot{I}_1 - \dot{I}_2 - \dot{I}_3 = 0$
- $\dot{I}_3 - \dot{I}_5 + \dot{J}_4 = 0$

Node-potential equation system

- $$\frac{-\dot{V}_A + \dot{E}_1}{R_1} - \frac{\dot{V}_A}{Z_{L2}} - \frac{\dot{V}_A - \dot{V}_B}{Z_{C3}} = 0$$
- $$\frac{\dot{V}_A - \dot{V}_B}{Z_{C3}} - \frac{\dot{V}_B}{R_5} + \dot{J}_4 = 0$$

Nodal Analysis

Matrix form of the node-potential equation system

$$\begin{bmatrix} \dfrac{1}{R_1} + \dfrac{1}{Z_{L_2}} + \dfrac{1}{Z_{C_3}} & -\dfrac{1}{Z_{C_3}} \\ -\dfrac{1}{Z_{C_3}} & \dfrac{1}{R_5} + \dfrac{1}{Z_{C_3}} \end{bmatrix} \begin{bmatrix} \dot{V}_A \\ \dot{V}_B \end{bmatrix} = \begin{bmatrix} \dfrac{\dot{E}_1}{R_1} \\ \dot{J}_4 \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} \dfrac{1}{5} + \dfrac{1}{j3} + \dfrac{1}{-j5} & -\dfrac{1}{-j5} \\ -\dfrac{1}{-j5} & \dfrac{1}{8} + \dfrac{1}{-j5} \end{bmatrix} \begin{bmatrix} \dot{V}_A \\ \dot{V}_B \end{bmatrix} = \begin{bmatrix} 2{,}4\underline{/20^\circ} \\ 1\underline{/-15^\circ} \end{bmatrix}$$

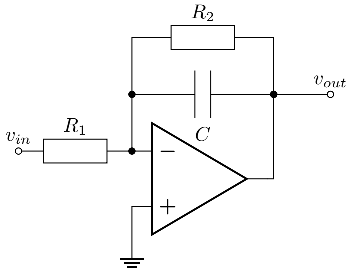
- Solve the system using algebraic methods

$\dot{V}_A = 8{,}0964\underline{/62{,}93^\circ}\text{ V}$

$\dot{V}_B = 2{,}8603\underline{/76{,}88^\circ}\text{ V}$

Nodal Analysis

Consider an op-amp circuit



Exercise:
Inspect the solution and comment on the circuit property.

Ideal op-amp

- Ideal input currents are zero

$\Rightarrow$ $\dot{I}^+ = 0;\ \dot{I}^- = 0$

- Ideal gain is ∞

$\Rightarrow$ $\dot{V}^+ = \dot{V}^-$

$$\dot{I}_{R1} - \dot{I}_{R2} - \dot{I}_C = 0$$

$$\Leftrightarrow \frac{\dot{V}_{in}}{R_1} + \frac{\dot{V}_{out}}{R_2} + \frac{\dot{V}_{out}}{Z_C} = 0$$

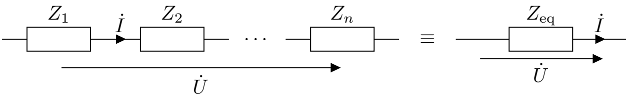
$$\Leftrightarrow \dot{V}_{out} = -\frac{\frac{1}{R_1}}{\frac{1}{R_2} + j\omega C}\dot{V}_{in}$$

Chapter Outline

5. Methods for Solving Linear Steady-State AC circuits
   - Branch-Current Method
   - Loop Analysis
   - Nodal Analysis
   - Equivalent-Impedance Method
   - Superposition Method
   - Power in Sinusoidal AC Circuits

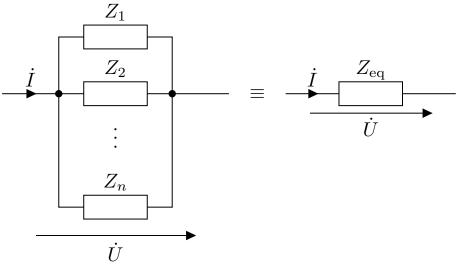
Equivalent-Impedance Method

Series circuit: equivalent impedance equals total impedance



$$Z_{eq} = Z_1 + Z_2 + \ldots + Z_n$$

Parallel circuit: equivalent admittance equals total admittance

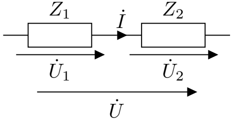


$$Y_{eq} = Y_1 + Y_2 + \ldots + Y_n$$

$$\Leftrightarrow \frac{1}{Z_{eq}} = \frac{1}{Z_1} + \frac{1}{Z_2} + \ldots + \frac{1}{Z_n}$$

Equivalent-Impedance Method

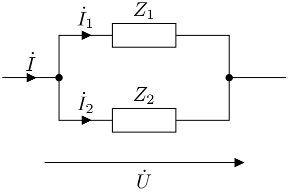
Voltage-division formula



- General formula: $\dot{U}_k = \dfrac{Z_k}{\sum_{i=1}^{n} Z_i} \cdot \dot{U}$
- Two-element case (common)

$$\dot{U}_1 = \frac{Z_1}{Z_1 + Z_2} \cdot \dot{U}; \quad \dot{U}_2 = \frac{Z_2}{Z_1 + Z_2} \cdot \dot{U}$$

Current-division formula

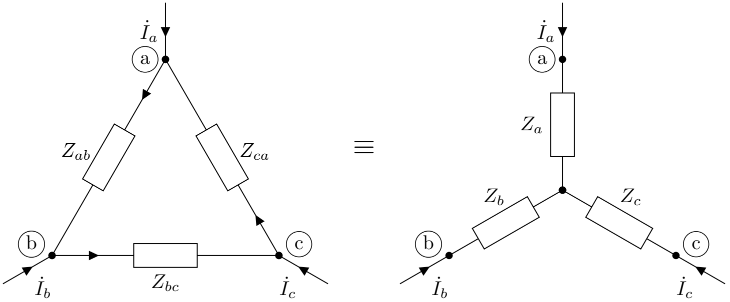


- General formula: $\dot{I}_k = \dfrac{Y_k}{\sum_{i=1}^{n} Y_i} \cdot \dot{I}$
- Two-element case (common)

$$\dot{I}_1 = \frac{Z_2}{Z_1 + Z_2} \cdot \dot{I}; \quad \dot{I}_2 = \frac{Z_1}{Z_1 + Z_2} \cdot \dot{I}$$

Equivalent-Impedance Method

Wye–delta transformation



$$\mathrm{Y} \to \Delta: \quad \begin{aligned} Z_{ab} &= Z_a + Z_b + \frac{Z_a \cdot Z_b}{Z_c} \\ Z_{bc} &= Z_b + Z_c + \frac{Z_b \cdot Z_c}{Z_a} \\ Z_{ca} &= Z_c + Z_a + \frac{Z_c \cdot Z_a}{Z_b} \end{aligned}$$

$$\Delta \to \mathrm{Y}: \quad \begin{aligned} Z_a &= \frac{Z_{ca} \cdot Z_{ab}}{Z_{ab} + Z_{bc} + Z_{ca}} \\ Z_b &= \frac{Z_{ab} \cdot Z_{bc}}{Z_{ab} + Z_{bc} + Z_{ca}} \\ Z_c &= \frac{Z_{bc} \cdot Z_{ca}}{Z_{ab} + Z_{bc} + Z_{ca}} \end{aligned}$$

Chapter Outline

5. Methods for Solving Linear Steady-State AC circuits
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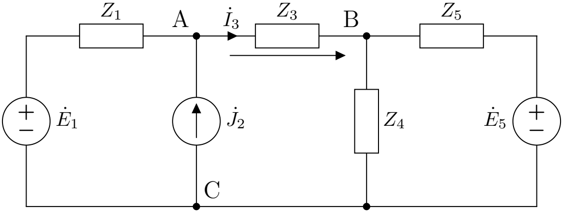
    - Superposition Method
    - Power in Sinusoidal AC Circuits

Superposition Method

Example 5.7: Determine current $\dot{I}_3$ using superposition

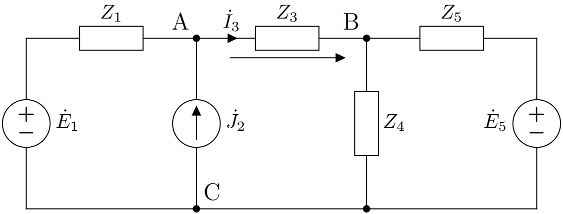


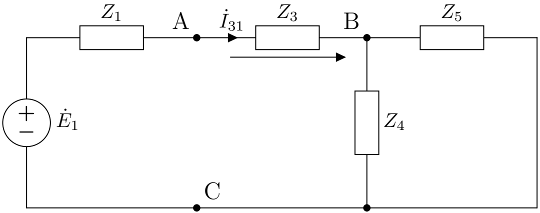
Solution

- $\dot{I}_3 = \dot{I}_{31} + \dot{I}_{32} + \dot{I}_{35}$
- $\dot{I}_{31}$ is the current through Z_3 when only $\dot{E}_1$ acts, with $\dot{J}_2$ and $\dot{E}_5$ off
- $\dot{I}_{32}$ is the current through Z_3 when only $\dot{J}_2$ acts, with $\dot{E}_1$ and $\dot{E}_5$ off
- $\dot{I}_{35}$ is the current through Z_3 when only $\dot{E}_5$ acts, with $\dot{J}_2$ and $\dot{E}_1$ off

Superposition Method

Example 5.7: Determine current $\dot{I}_3$ using superposition





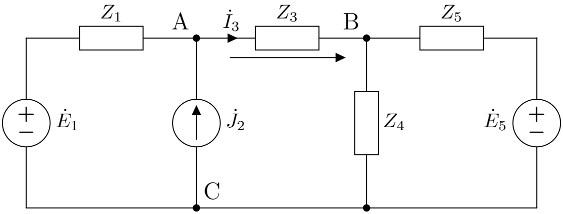
Component $\dot{I}_{31}$

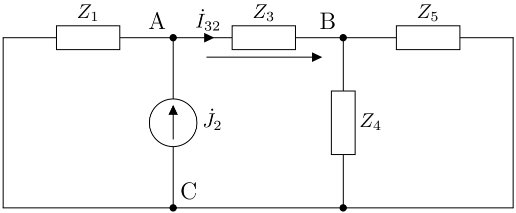
$$Z_{\mathbf{eq}} = Z_1 + Z_3 + \frac{Z_4 \cdot Z_5}{Z_4 + Z_5}$$

$$\dot{I}_{31} = \frac{\dot{E}_1}{Z_{\mathbf{eq}}}$$

Superposition Method

Example 5.7: Determine current $\dot{I}_3$ using superposition





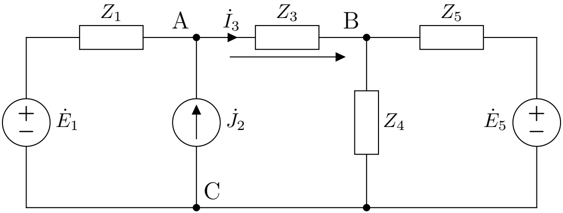
Component $\dot{I}_{32}$

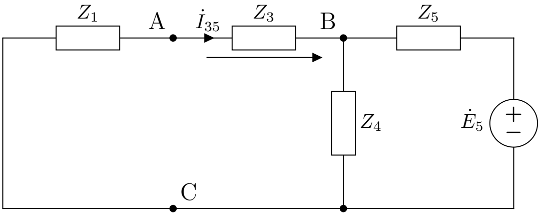
$$Z_{345} = Z_3 + \frac{Z_4 \cdot Z_5}{Z_4 + Z_5}$$

$$\dot{I}_{32} = \frac{Z_1}{Z_1 + Z_{345}} \cdot \dot{J}_2$$

Superposition Method

Example 5.7: Determine current $\dot{I}_3$ using superposition





Component $\dot{I}_{35}$

$$Z_{\mathbf{eq}} = Z_5 + \frac{Z_4 \cdot (Z_1 + Z_3)}{Z_1 + Z_3 + Z_4}$$

$$\dot{I}_{32} = -\frac{\dot{E}_5}{Z_{\mathbf{eq}}} \cdot \frac{Z_4}{Z_1 + Z_3 + Z_4}$$

Chapter Outline

5. Methods for Solving Linear Steady-State AC circuits
   - Branch-Current Method
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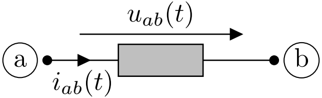
   - Power in Sinusoidal AC Circuits

Instantaneous Power Concept



- Instantaneous absorbed power: $p(t) = u_{ab}(t) \cdot i_{ab}(t)$
- When current and voltage are sinusoidal functions

$u_{ab}(t) = U_0 \sin(\omega t + \varphi)$

$i_{ab}(t) = I_0 \sin(\omega t + \theta)$

$$\Rightarrow p(t) = U_0 I_0 \sin(\omega t + \varphi) \sin(\omega t + \theta)$$

$$= \frac{U_0 I_0}{2} \left[\cos(\varphi - \theta) - \cos(2\omega t + \varphi + \theta)\right]$$

$$= \underbrace{\frac{U_0 I_0}{2} \cos(\varphi - \theta)}_{\text{Constant component}} - \underbrace{\frac{U_0 I_0}{2} \cos(2\omega t + \varphi + \theta)}_{\text{Time-varying component}}$$

Average Power Concept

- Average power P is the average value of instantaneous power $p(t)$

$$
\begin{aligned}
P &= \frac{1}{T}\int_0^T p(t)\cdot dt \\
&= \frac{1}{T}\int_0^T \left[\underbrace{\frac{U_0 I_0}{2}\cos\left(\varphi-\theta\right)}_{\text{Constant component}} - \underbrace{\frac{U_0 I_0}{2}\cos\left(2\omega t+\varphi+\theta\right)}_{\text{Time-varying component}}\right] dt \\
&= \frac{U_0 I_0}{2}\cos\left(\varphi-\theta\right) \\
&= \frac{U_0}{\sqrt{2}}\cdot\frac{I_0}{\sqrt{2}}\cdot\cos\left(\varphi-\theta\right)
\end{aligned}
$$

$\Rightarrow$ Average power on a resistor equals the product of the RMS current and voltage (always ≥ 0)

$\Rightarrow$ Average power on an inductor and a capacitor is 0

Average Power Concept

Computing average power from current and voltage phasors

- $u_{ab}(t) = U_0 \sin(\omega t + \varphi) \to \dot{U} = \frac{U_0}{\sqrt{2}} \angle \varphi$
- $i_{ab}(t) = I_0 \sin(\omega t + \theta) \to \dot{I} = \frac{I_0}{\sqrt{2}} \angle \theta \to \dot{I}^* = \frac{I_0}{\sqrt{2}} \angle -\theta$

$$\Rightarrow \dot{U} \cdot \dot{I}^* = \frac{U_0 I_0}{2} \angle \varphi - \theta$$

$$= \frac{U_0 I_0}{2} \left(\cos(\varphi - \theta) - j \sin(\varphi - \theta)\right)$$

$$= \frac{U_0}{\sqrt{2}} \cdot \frac{I_0}{\sqrt{2}} \cos(\varphi - \theta) - j \cdot \frac{U_0}{\sqrt{2}} \cdot \frac{I_0}{\sqrt{2}} \sin(\varphi - \theta)$$

$\Rightarrow P = \mathrm{Re}\left(\dot{U} \cdot \dot{I}^*\right)$ (W) $\to$ also called real power (or active power)

---

x^*: complex conjugate of x

Extending the Power Concept

- The phasor-domain viewpoint allows us to extend the concept of power

Apparent power S

- The "visible" power obtained by multiplying the RMS current and voltage

$$S = \frac{U_0}{\sqrt{2}} \cdot \frac{I_0}{\sqrt{2}} \,(\mathrm{VA})$$

$\Rightarrow$ S is the modulus of $\left(\dot{U} \cdot \dot{I}^*\right)$; its unit is volt-ampere (NOT watt)

Reactive power Q

- Real power P is always less than apparent power S

$\Rightarrow$ There is another component: the imaginary part of the product $\left(\dot{U} \cdot \dot{I}^*\right)$

$\Rightarrow$ $Q = \mathrm{Im}\left(\dot{U} \cdot \dot{I}^*\right)$ (VAR) (volt-ampere-reactive)

Extending the Power Concept

Complex power

$$\dot{S} = \left(\dot{U} \cdot \dot{I}^*\right)$$

$\Rightarrow$ It contains active power in the real part and reactive power in the imaginary part: $\dot{S} = P + j \cdot Q$; its unit is VA

$\Rightarrow$ Apparent power is the modulus of complex power: $S = |\dot{S}| = \sqrt{P^2 + Q^2}$

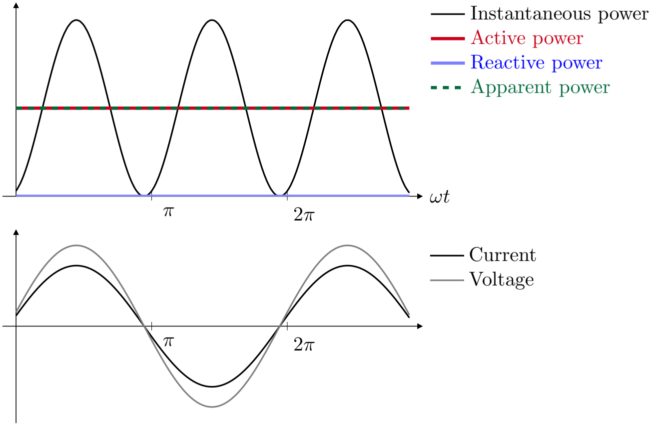
Power factor

- The ratio between real power and apparent power: $\text{pf} = \dfrac{P}{S} = \cos\left(\varphi - \theta\right)$
- Often called the $\cos\varphi$ coefficient

Summary of Power Concepts

Purely resistive load: current and voltage are in phase

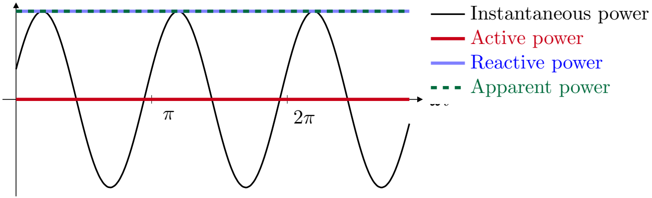
- $\varphi = \theta \to$ power factor $\cos(\varphi - \theta) = 1$

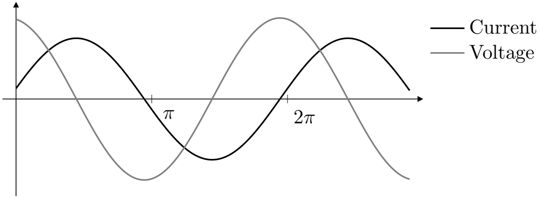


Summary of Power Concepts

Purely inductive load: current lags voltage by 90°

- $\varphi - \theta = 90^\circ \rightarrow$ power factor $\cos(\varphi - \theta) = 0$

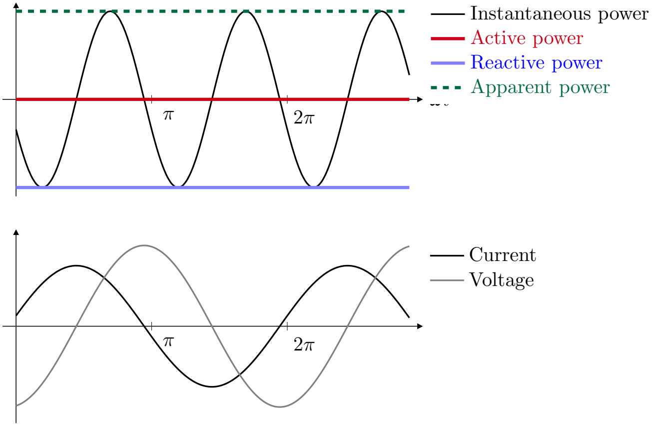




Summary of Power Concepts

Purely capacitive load: voltage lags current by 90°

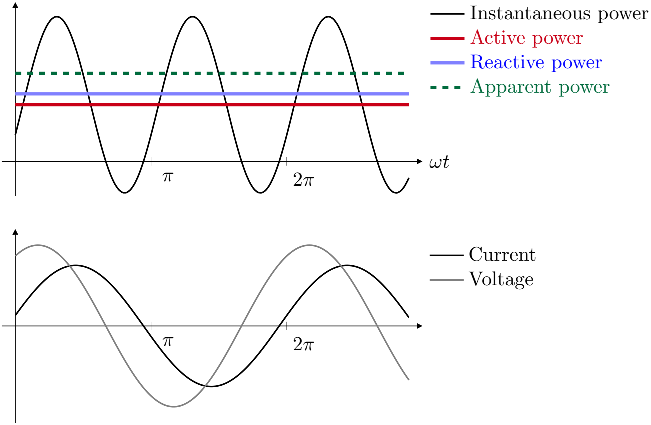
- $\varphi - \theta = -90° \rightarrow$ power factor $\cos(\varphi - \theta) = 0$



Summary of Power Concepts

Inductive load: current lags voltage

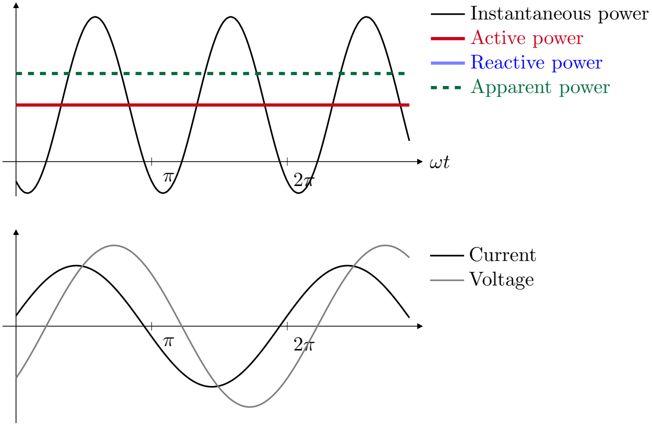
- $\varphi > \theta \to$ power factor $\cos(\varphi - \theta) < 1$



Summary of Power Concepts

Capacitive load: voltage lags current

- $\varphi < \theta \rightarrow$ power factor $\cos(\varphi - \theta) < 1$



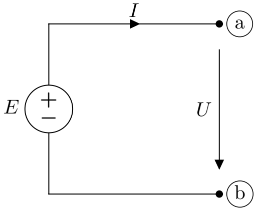
Course Outline

Chapter Outline

Chapter Outline

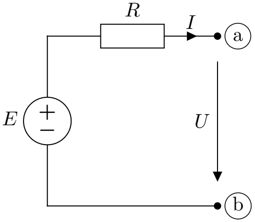
Practical Voltage-Source and Current-Source Models

Ideal DC voltage source



$$U = E$$

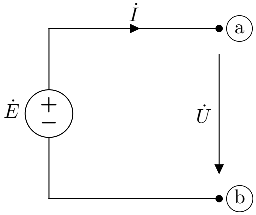
Practical DC voltage source



$$U = E - R \cdot I$$

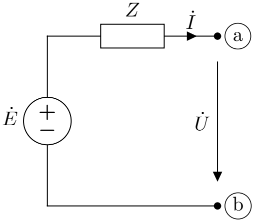
Practical Voltage-Source and Current-Source Models

Ideal AC voltage source



$$\dot{U} = \dot{E}$$

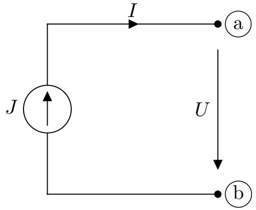
Practical AC voltage source



$$\dot{U} = \dot{E} - Z \cdot \dot{I}$$

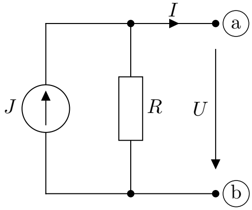
Practical Voltage-Source and Current-Source Models

Ideal DC current source



$$I = J$$

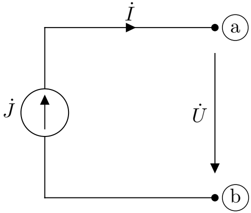
Practical DC current source



$$I = J - \frac{U}{R} \Leftrightarrow U = R \cdot J - R \cdot I$$

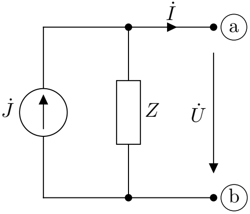
Practical Voltage-Source and Current-Source Models

Ideal AC current source



$$\dot{I} = \dot{J}$$

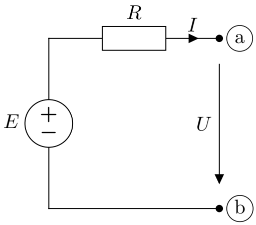
Practical AC current source



$$\dot{I} = \dot{J} - \frac{\dot{U}}{Z} \Leftrightarrow \dot{U} = Z \cdot \dot{J} - Z \cdot \dot{I}$$

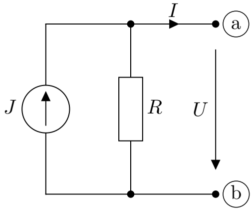
Source Transformation

Practical DC voltage source



$$U = E - R \cdot I$$

Practical DC current source

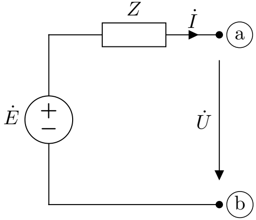


$$U = R \cdot J - R \cdot I$$

$\Rightarrow$ If $E = R \cdot J \Leftrightarrow J = \dfrac{E}{R}$, then **the two models are equivalent**

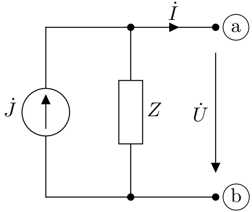
Source Transformation

Practical AC voltage source



$$\dot{U} = \dot{E} - Z \cdot \dot{I}$$

Practical AC current source



$$\dot{U} = Z \cdot \dot{J} - Z \cdot \dot{I}$$

$\Rightarrow$ If $\dot{E} = Z \cdot \dot{J} \Leftrightarrow \dot{J} = \dfrac{\dot{E}}{Z}$, then **the two models are equivalent**

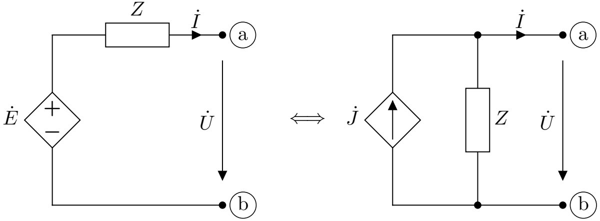
$\Rightarrow$ Source transformation method

If a circuit contains a practical voltage-source and/or current-source model, source transformation can be used to simplify the circuit structure.

Source Transformation

Some notes

- The source-transformation formulas for DC circuits and AC circuits in the complex domain have equivalent forms $\Rightarrow$ the lecture may discuss only one form
- Source transformation can be applied to dependent sources, provided that $E = R \cdot J \Leftrightarrow J = \dfrac{E}{R}$ or $\dot{E} = Z \cdot \dot{J} \Leftrightarrow \dot{J} = \dfrac{\dot{E}}{Z}$ is satisfied

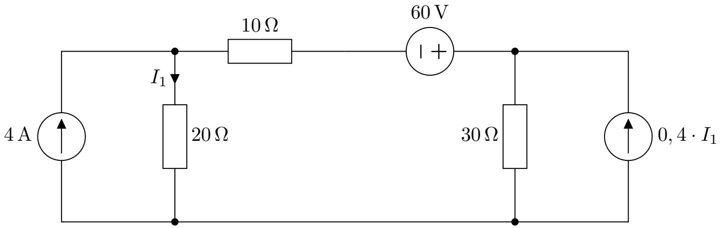


Source Transformation

Some notes

- Source transformation changes the circuit structure
  $\Rightarrow$ do not transform the element containing the current or voltage to be found
- Directions of current sources, voltage sources, and equivalent currents and voltages

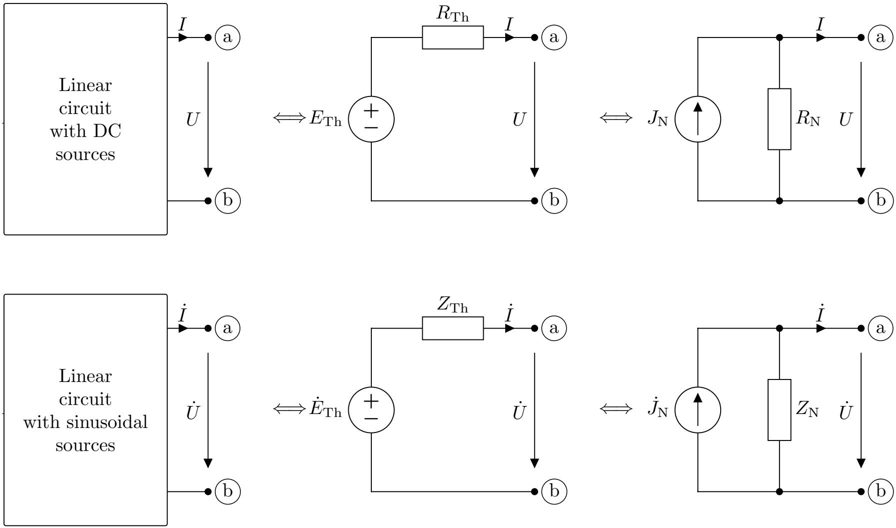
Example 6.1: Determine I_1 using source transformation



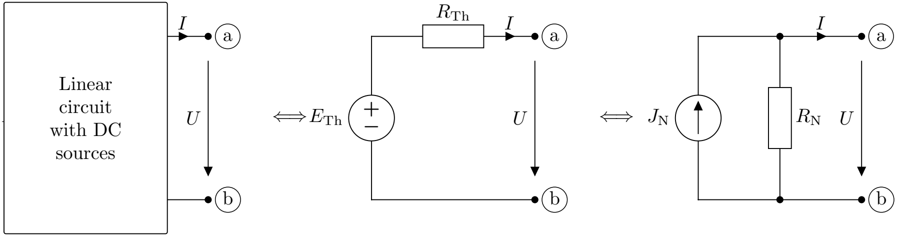
Chapter Outline

6. Thévenin–Norton Theorem and One-Port Networks
   - Source Transformation Method
   - Thévenin–Norton Theorem for One-Port Network Models
   - Impedance Matching Problem

Thévenin–Norton Theorem for One-Port Network



Determining Thévenin–Norton Parameters

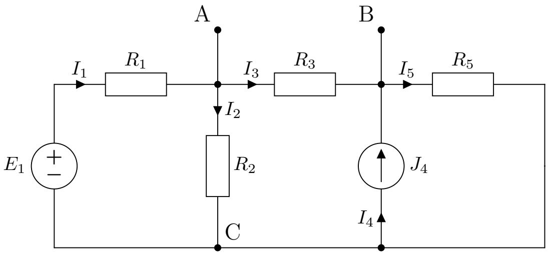


- The equivalent voltage source equals the open-circuit voltage: $E_{\mathrm{Th}} = U_{\mathrm{oc}}$
- The equivalent current source equals the short-circuit current: $J_{\mathrm{N}} = I_{\mathrm{sc}}$
- The resistance $R_{\mathrm{Th}} = R_{\mathrm{N}}$ can be determined in 2 ways:
  1. $R_{\mathrm{Th}} = R_{\mathrm{N}} = \dfrac{E_{\mathrm{Th}}}{I_{\mathrm{N}}} = \dfrac{E_{\mathrm{oc}}}{I_{\mathrm{sc}}}$
  2. $R_{\mathrm{Th}} = R_{\mathrm{N}} = R_{\mathrm{eq}}$ when the independent sources are suppressed
- The same applies to AC circuits

Determining Thévenin–Norton Parameters

Example 6.2

- Determine the Thévenin one-port model at nodes A and B
- Determine the Norton equivalent one-port model at nodes A and B
- Use the Thévenin–Norton theorem to determine I_3



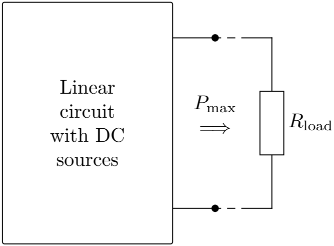
- $E_1 = 15\,\mathrm{V}$; $R_1 = 5\,\Omega$;
- $R_2 = 8\,\Omega$; $R_3 = 6\,\Omega$;
- $J_4 = 1\,\mathrm{A}$; $R_5 = 10\,\Omega$.

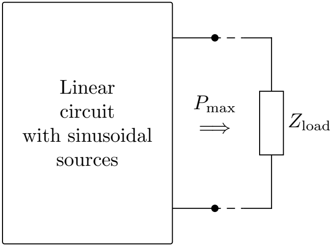
Chapter Outline

Impedance Matching Problem

Problem statement

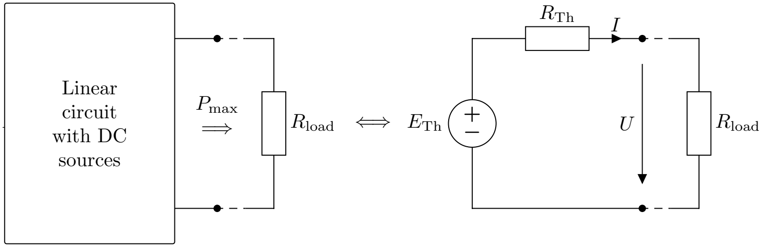
- Determine the load value that maximizes the power received from the source circuit





Impedance Matching Problem

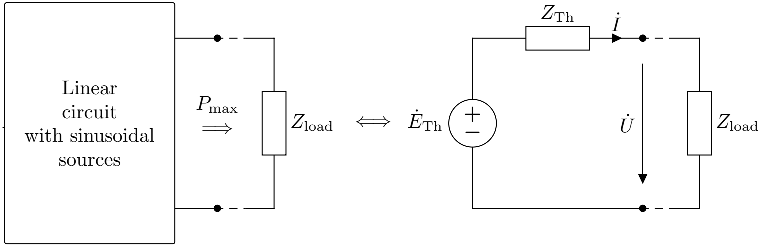
Solution: Use the Thévenin–Norton theorem



- $I = \dfrac{E_{Th}}{R_{Th} + R_l} \Leftrightarrow P = R_l \cdot I^2 = \dfrac{R_l \cdot E_{Th}^2}{(R_{Th} + R_l)^2} \leq \dfrac{E_{Th}^2}{4R_{Th}}$
- P is maximized when the equality sign "=" occurs, with $R_l = R_{Th}$
- Prove this with the Norton one-port model?

Impedance Matching Problem

Solution: Use the Thévenin–Norton theorem



- $\dot{I} = \dfrac{\dot{E}_{\text{Th}}}{Z_{\text{Th}} + Z_{\text{l}}} \Leftrightarrow P = \text{Re}(Z_{\text{l}}) \cdot |\dot{I}|^2$

$$= \frac{\text{Re}(Z_{\text{l}}) \cdot |\dot{E}_{\text{Th}}|^2}{(\text{Re}(Z_{\text{Th}}) + \text{Re}(Z_{\text{l}}))^2 + (\text{Im}(Z_{\text{Th}}) + \text{Im}(Z_{\text{l}}))^2} \le \frac{|\dot{E}_{\text{Th}}|^2}{4\,\text{Re}(Z_{\text{Th}})}$$

- P is maximized when the equality sign "=" occurs, with $Z_{\text{l}} = Z_{\text{Th}}^*$

Impedance Matching Problem

Example 6.3

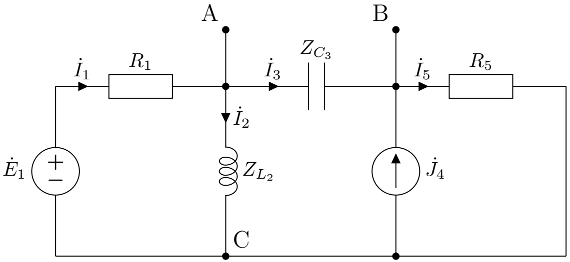
- Determine the matched load to connect between nodes A and B for maximum power transfer

$\dot{E}_1 = 12\underline{/20^\circ}\,\mathrm{V}$;

$R_1 = 5\,\Omega$; $Z_{L_2} = j3\,\Omega$;

$Z_{C_3} = -j5\,\Omega$; $R_5 = 8\,\Omega$;

$\dot{J}_4 = 1\underline{/-15^\circ}\,\mathrm{A}$.



Course Outline

Chapter Outline

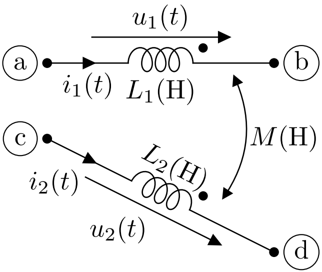
7. Mutual Inductance and Solving Magnetically Coupled Circuits
   - Mutual-Inductance Phenomenon
   - Methods for Solving Magnetically Coupled Circuits
   - Mutual-Inductance Power
   - One-Port Networks Containing Mutual Inductance

Chapter Outline

7. Mutual Inductance and Solving Magnetically Coupled Circuits
   - Mutual-Inductance Phenomenon
   - Methods for Solving Magnetically Coupled Circuits
   - Mutual-Inductance Power
   - One-Port Networks Containing Mutual Inductance

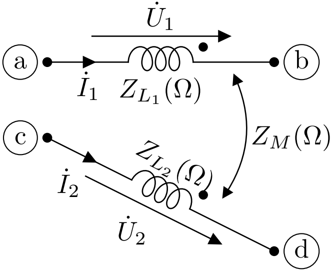
Mutual Inductance Between Two Coils

Mutual inductance in the time domain



$$u_1(t) = L_1 \cdot \frac{di_1(t)}{dt} \pm M \cdot \frac{di_2(t)}{dt}$$
$$u_2(t) = L_2 \cdot \frac{di_2(t)}{dt} \pm M \cdot \frac{di_1(t)}{dt}$$

Phasor representation of mutual inductance



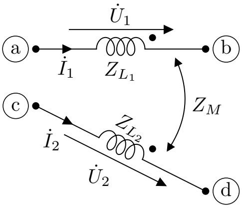
$$\begin{aligned}\dot{U}_1 &= j\omega L_1 \cdot \dot{I}_1 \pm j\omega M \cdot \dot{I}_2 \\ &= Z_{L_1} \cdot \dot{I}_1 \pm Z_M \cdot \dot{I}_2 \\ \dot{U}_2 &= j\omega L_2 \cdot \dot{I}_2 \pm j\omega M \cdot \dot{I}_1 \\ &= Z_{L_2} \cdot \dot{I}_2 \pm Z_M \cdot \dot{I}_1\end{aligned}$$

Sign of the Mutual-Inductance Electromotive Force

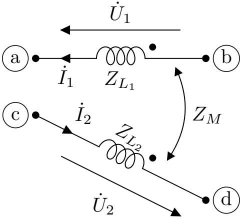
Rule for determining the sign of the mutual-inductance component

- If the directions of both currents both enter or both leave the polarity dots, the mutual-inductance component M has the same sign as the self-inductance component L of the coil being considered
- If one current enters and one current leaves the polarity dots, the mutual-inductance component M has the opposite sign to the self-inductance component L of the coil being considered

- Same sign



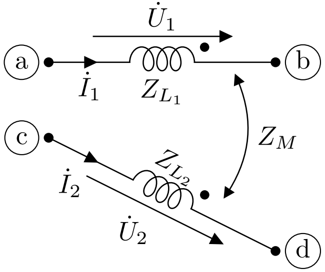
- Opposite sign



Mutual-Inductance Model Using Dependent Source

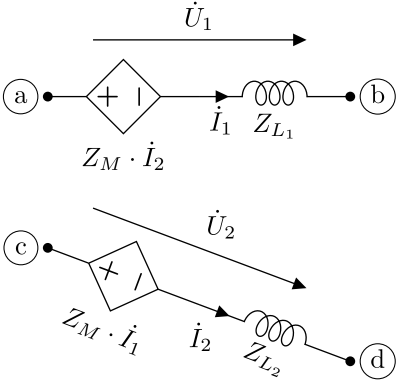
- Same-sign mutual inductance: Example 1

Mutual inductance



$$\dot{U}_1 = Z_{L_1} \cdot \dot{I}_1 + Z_M \cdot \dot{I}_2$$
$$\dot{U}_2 = Z_{L_2} \cdot \dot{I}_2 + Z_M \cdot \dot{I}_1$$

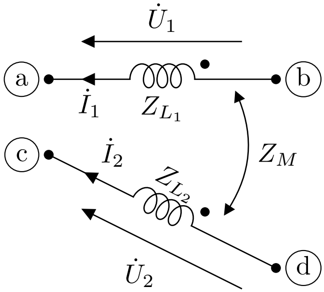
Equivalent model



Mutual-Inductance Model Using Dependent Source

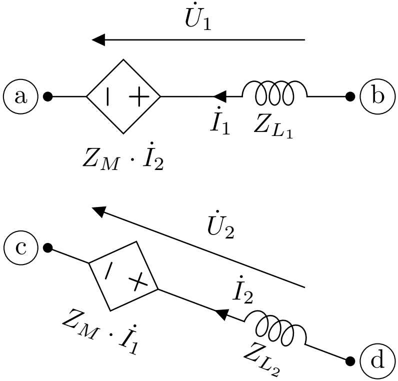
- Same-sign mutual inductance: Example 2

Mutual inductance



$$\dot{U}_1 = Z_{L_1} \cdot \dot{I}_1 + Z_M \cdot \dot{I}_2$$
$$\dot{U}_2 = Z_{L_2} \cdot \dot{I}_2 + Z_M \cdot \dot{I}_1$$

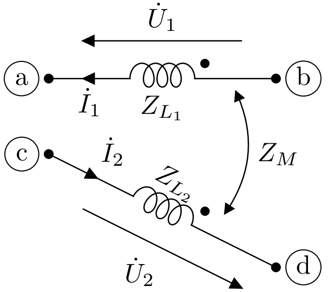
Equivalent model



Mutual-Inductance Model Using Dependent Source

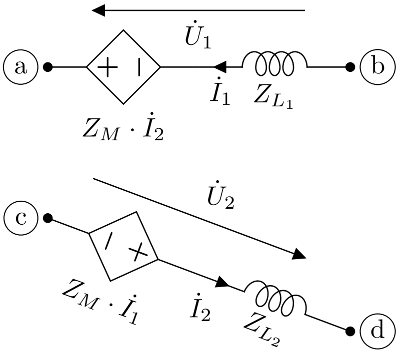
- Opposite-sign mutual inductance: Example 1

Mutual inductance



$$\dot{U}_1 = Z_{L_1} \cdot \dot{I}_1 - Z_M \cdot \dot{I}_2$$
$$\dot{U}_2 = Z_{L_2} \cdot \dot{I}_2 - Z_M \cdot \dot{I}_1$$

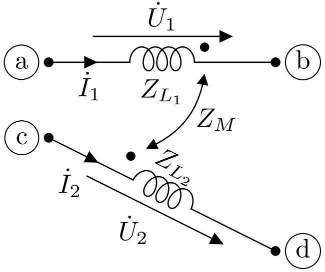
Equivalent model



Mutual-Inductance Model Using Dependent Source

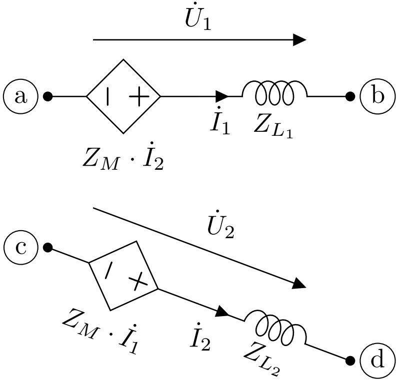
- Opposite-sign mutual inductance: Example 2

Mutual inductance



$$\dot{U}_1 = Z_{L_1} \cdot \dot{I}_1 - Z_M \cdot \dot{I}_2$$
$$\dot{U}_2 = Z_{L_2} \cdot \dot{I}_2 - Z_M \cdot \dot{I}_1$$

Equivalent model



Chapter Outline

Methods for Solving Magnetically Coupled Circuits

Steps for solving circuits with mutual inductance

1. Determine the sign of the mutual inductance
2. Convert the circuit with mutual inductance into equivalent model
3. Solve the circuit using studied methods

Some notes

⚠ Because a magnetically coupled circuit is equivalent to current-controlled voltage sources, some methods are affected

- Equivalent impedance: cannot be determined directly
- Voltage-division and current-division: cannot be used directly
- The nodal analysis is inconvenient

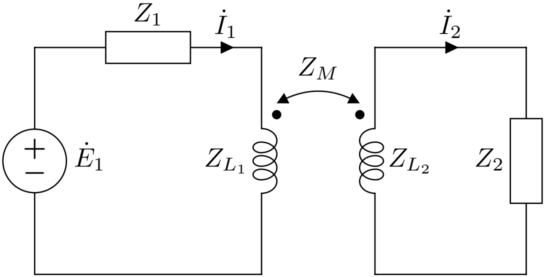
- Branch-current and loop analysis can be used normally

Methods for Solving Magnetically Coupled Circuits

Example 7.1

Given a transformer circuit. Determine currents and voltages on the coils.

$\dot{E}_1 = 12\underline{/0^\circ}$ V;
$Z_1 = 5 + j3\ (\Omega)$;
$Z_{L_1} = j8\ (\Omega)$;
$Z_{L_2} = j6\ (\Omega)$;
$Z_M = j6\ (\Omega)$;
$Z_2 = 5 - j2\ (\Omega)$



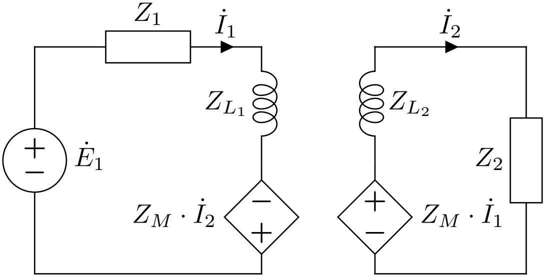
- Determine the sign of the mutual inductance: the mutual-inductance EMF (the M component) has the opposite sign to the self-inductance EMF (the L component)

Methods for Solving Magnetically Coupled Circuits

Example 7.1

Convert to an equivalent CCVS model

$\dot{E}_1 = 12\underline{/0^\circ}\,\text{V};$
$Z_1 = 5 + j3\ (\Omega);$
$Z_{L_1} = j8\ (\Omega);$
$Z_{L_2} = j6\ (\Omega);$
$Z_M = j6\ (\Omega);$
$Z_2 = 5 - j2\ (\Omega)$



- Number of branches: 2
- Number of essential nodes: 0

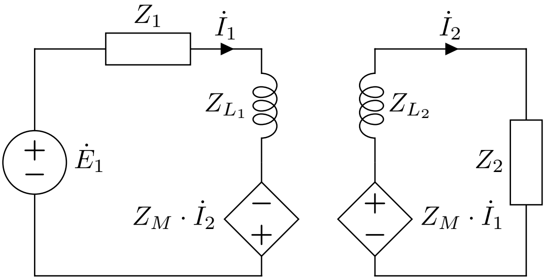
⇒ Number of loops: 2

☺ This circuit has the same number of branches and loops

Methods for Solving Magnetically Coupled Circuits

Example 7.1

$$\dot{E}_1 = 12\underline{/0^\circ}\ \text{V};$$
$$Z_1 = 5 + j3\ (\Omega);$$
$$Z_{L_1} = j8\ (\Omega);$$
$$Z_{L_2} = j6\ (\Omega);$$
$$Z_M = j6\ (\Omega);$$
$$Z_2 = 5 - j2\ (\Omega)$$



Loop-current (branch-current) equation system

$$\left(Z_1 + Z_{L_1}\right)\dot{I}_1 - Z_M\dot{I}_2 = \dot{E}_1$$
$$-Z_M\dot{I}_1 + \left(Z_2 + Z_{L_2}\right)\dot{I}_2 = 0$$

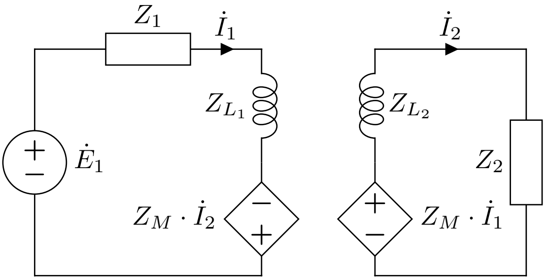
Substitute values

$$\left(5 + j11\right)\dot{I}_1 - j6\dot{I}_2 = 12\underline{/0^\circ}$$
$$-j6\dot{I}_1 + \left(5 + j4\right)\dot{I}_2 = 0$$

Methods for Solving Magnetically Coupled Circuits

Example 7.1

$\dot{E}_1 = 12\underline{/0^\circ}\,\text{V};$
$Z_1 = 5 + j3\ (\Omega);$
$Z_{L_1} = j8\ (\Omega);$
$Z_{L_2} = j6\ (\Omega);$
$Z_M = j6\ (\Omega);$
$Z_2 = 5 - j2\ (\Omega)$



Currents

$\dot{I}_1 = 0{,}9992\underline{/-38{,}57^\circ}\ (\text{A})$

$\dot{I}_2 = 0{,}9363\underline{/12{,}77^\circ}\ (\text{A})$

Voltages on the coils

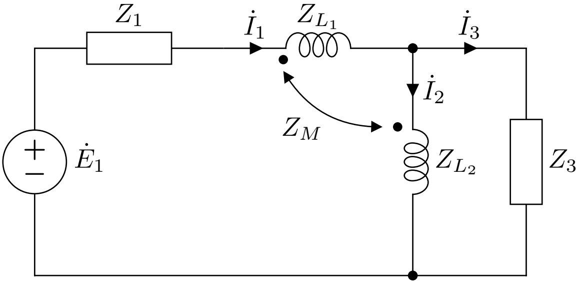
$\dot{U}_{L_1} = Z_{L_1}\dot{I}_1 - Z_M\dot{I}_2 = 6{,}2728\underline{/7{,}06^\circ}\ (\text{V})$

$\dot{U}_{L_2} = Z_{L_2}\dot{I}_2 - Z_M\dot{I}_1 = 5{,}0419\underline{/170{,}97^\circ}\ (\text{V})$

Methods for Solving Magnetically Coupled Circuits

Example 7.2

$\dot{E}_1 = 12\angle 0^\circ$ V;
$Z_1 = 5 + j3\ (\Omega)$;
$Z_{L_1} = j8\ (\Omega)$;
$Z_{L_2} = j6\ (\Omega)$;
$Z_M = j6\ (\Omega)$;
$Z_3 = 5 - j2\ (\Omega)$



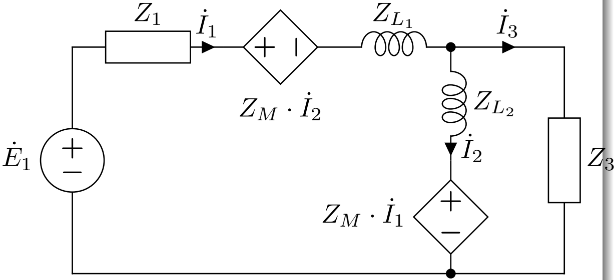
- Determine the sign of the mutual inductance: the mutual-inductance EMF (the M component) has the same sign as the self-inductance EMF (the L component)

Methods for Solving Magnetically Coupled Circuits

Example 7.2

Convert to an equivalent CCVS model

$\dot{E}_1 = 12\underline{/0^\circ}$ V;
$Z_1 = 5 + j3\ (\Omega)$;
$Z_{L_1} = j8\ (\Omega)$;
$Z_{L_2} = j6\ (\Omega)$;
$Z_M = j6\ (\Omega)$;
$Z_3 = 5 - j2\ (\Omega)$



Loop-current equation system

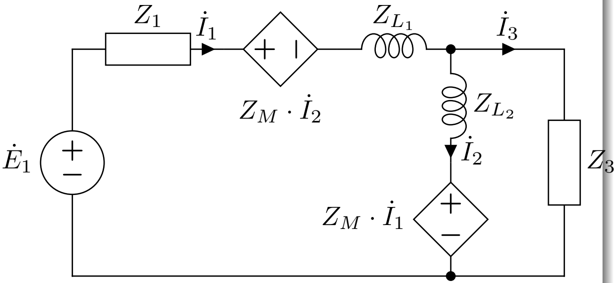
$$(Z_1 + Z_{L_1} + Z_{L_2} + 2Z_M)\dot{I}_{V1} - (Z_{L_2} + Z_M)\dot{I}_{V2} = \dot{E}_1$$
$$-(Z_{L_2} + Z_M)\dot{I}_{V1} + (Z_3 + Z_{L_2})\dot{I}_{V2} = 0$$

Methods for Solving Magnetically Coupled Circuits

Example 7.2

Convert to an equivalent CCVS model

$\dot{E}_1 = 12\underline{/0^\circ}$ V;
$Z_1 = 5 + j3$ (Ω);
$Z_{L_1} = j8$ (Ω);
$Z_{L_2} = j6$ (Ω);
$Z_M = j6$ (Ω);
$Z_3 = 5 - j2$ (Ω)



Substitute values

$$(5 + j29)\,\dot{I}_{V1} - j12\dot{I}_{V2} = 12\underline{/0^\circ}$$
$$-j12\dot{I}_{V1} + (5 + j4)\,\dot{I}_{V2} = 0$$

⇒ Loop currents

$$\dot{I}_{V1} = 0{,}4434\underline{/-33{,}53^\circ} \text{ (A)}$$
$$\dot{I}_{V2} = 0{,}8309\underline{/17{,}81^\circ} \text{ (A)}$$

Chapter Outline

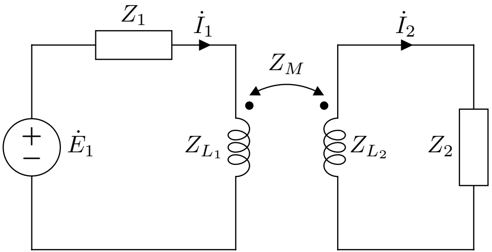
7 Mutual Inductance and Solving Magnetically Coupled Circuits

- Mutual-Inductance Phenomenon
- Methods for Solving Magnetically Coupled Circuits
- Mutual-Inductance Power
- One-Port Networks Containing Mutual Inductance

Mutual-Inductance Power

Return to Example 7.1

$$\dot{I}_1 = 0{,}9992\underline{/-38{,}57^\circ}\ \text{(A)}$$
$$\dot{I}_2 = 0{,}9363\underline{/12{,}77^\circ}\ \text{(A)}$$
$$\dot{U}_{L_1} = 6{,}2728\underline{/7{,}06^\circ}\ \text{(V)}$$
$$\dot{U}_{L_2} = 5{,}0419\underline{/170{,}97^\circ}\ \text{(V)}$$



Observation: The current and voltage on the coils are not in quadrature

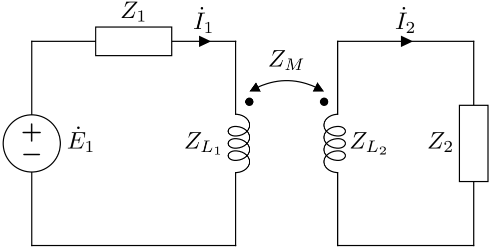
$\Rightarrow$ There is "absorbed" power on magnetically coupled coils

$$P_{L_1} = |\dot{U}_1| \cdot |\dot{I}_1| \cdot \cos\left(\underline{/\dot{U}_1} - \underline{/\dot{I}_1}\right) = 4{,}3828\ \text{(W)}$$
$$P_{L_2} = |\dot{U}_2| \cdot |\dot{I}_2| \cdot \cos\left(\underline{/\dot{U}_2} - \underline{/\dot{I}_2}\right) = -4{,}3828\ \text{(W)}$$

Mutual-Inductance Power

Return to Example 7.1

$$P_{L_1} = 4{,}3828 \text{ (W)}$$
$$P_{L_2} = -4{,}3828 \text{ (W)}$$
$$P_{Z_2} = -P_{L_2}$$



⇒ Observation

- The total absorbed power on the two coils is $P_{L_1} + P_{L_2} = 0$
- Mutual-inductance power is essentially the transfer of energy as a magnetic field between coils at different positions in the circuit
- The coil with $P > 0$ is primary; the coil with $P < 0$ is secondary

Chapter Outline

7. Mutual Inductance and Solving Magnetically Coupled Circuits
   - Mutual-Inductance Phenomenon
   - Methods for Solving Magnetically Coupled Circuits
   - Mutual-Inductance Power
   - One-Port Networks Containing Mutual Inductance

One-Port Networks Containing Mutual Inductance

- A linear circuit with mutual inductance is still a linear circuit $\Rightarrow$ the Thévenin–Norton theorems still hold
- Branches with mutual inductance are converted into equivalent diagrams containing dependent sources

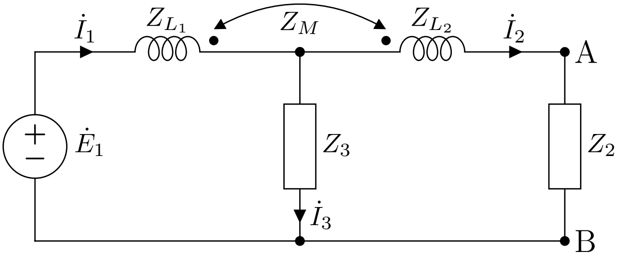
$\Rightarrow$ When computing equivalent impedance, use one of two "indirect" methods (which give the same result):

1. $Z_{\text{eq}} = \dfrac{\dot{U}_{\text{oc}}}{\dot{I}_{\text{sc}}} = \dfrac{\dot{E}_{\text{Th}}}{\dot{J}_{\text{N}}}$
2. Add a source at the port (or determine the transfer function of voltage and current at the port): $Z_{\text{eq}} = \dfrac{\dot{U}_{\text{port}}}{\dot{I}_{\text{port}}}$

One-Port Networks Containing Mutual Inductance

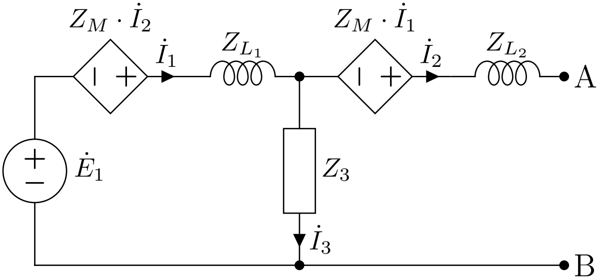
Example 7.3

Find Z_2 for impedance matching at port AB



One-Port Networks Containing Mutual Inductance

⇒ Equivalent model using dependent sources



- Determine $\dot{U}_{\text{oc}}$
- Determine $\dot{I}_{\text{sc}}$
- Determine $Z_{\text{eq}} = \dfrac{\dot{U}_{\text{oc}}}{\dot{I}_{\text{sc}}}$ ⇒ impedance matching when $Z_2 = Z_{\text{eq}}^*$

Course Outline

Chapter Outline

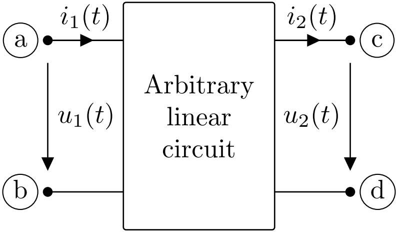
8. Two-Port Networks
   - Two-Port Network Models
   - Methods for Determining Two-Port Network Parameters
   - Methods for Solving Circuits with Two-Port Networks
   - Thévenin–Norton for Circuits Containing Two-Port Networks
   - Transfer Function Concept in Electric Circuits
   - Interconnection of Two-Port Networks

Chapter Outline

Two-Port Network Concept

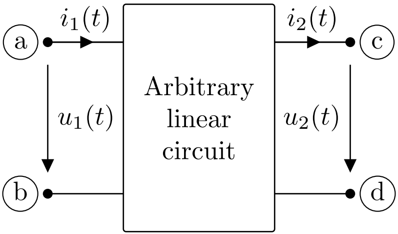
- In many cases, a circuit needs to be divided into connected *subcircuits*.
- These subcircuits are called **networks**, connected through **pairs of nodes**
- Each such *node* is called a **terminal**; a *pair of nodes* forms a **port**
- Networks (subcircuits) interact through *state-variable pairs* of current and voltage at the ports
- *One-port networks* were studied in Chapter 6
- This chapter considers **two-port networks**
- The concept of a two-port network can be extended to multi-port networks

Two-Port Network Concept



- A two-port network has 2 state-variable pairs: u_1, i_1 and u_2, i_2
- At each port, the current entering one terminal equals the current leaving the other terminal
- Pay attention to the reference directions of current and voltage
- Within the scope of Circuit Theory, we consider only two-port networks without independent sources (but dependent sources)
- The equations for DC and AC two-port networks have similar forms → the lecture gives equations for only one type

Characteristic Relations of a Two-Port Network



- Relation of u_1, i_1 in terms of u_2, i_2 → matrix **A** (forward transmission)
- Relation of u_2, i_2 in terms of u_1, i_1 → matrix **B** (reverse transmission)
- Relation of u_1, u_2 in terms of i_1, i_2 → matrix **Z** (impedance)
- Relation of i_1, i_2 in terms of u_1, u_2 → matrix **Y** (admittance)
- Relation of u_1, i_2 in terms of i_1, u_2 → matrix **H** (forward hybrid)
- Relation of i_1, u_2 in terms of u_1, i_2 → matrix **G** (reverse hybrid)

Parameter Matrices of a Two-Port Network

- They are square matrices of size $2 \times 2 \rightarrow$ each matrix has 4 parameters to determine
- Circuit Theory often focuses on matrices **A**, **Z**, and **Y** and they match commonly used circuit-solving methods:
  - Matrix **A** is convenient for computing equivalent impedance
  - Matrix **Z** is convenient for the loop analysis
  - Matrix **Y** is convenient for the nodal analysis
  - Note: convenient, but not mandatory
- Example: matrix **A**

$$\begin{cases} \dot{U}_1 = a_{11} \cdot \dot{U}_2 + a_{12} \cdot \dot{I}_2 \\ \dot{I}_1 = a_{21} \cdot \dot{U}_2 + a_{22} \cdot \dot{I}_2 \end{cases}$$

$$\Leftrightarrow \begin{bmatrix} \dot{U}_1 \\ \dot{I}_1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \cdot \begin{bmatrix} \dot{U}_2 \\ \dot{I}_2 \end{bmatrix} \Leftrightarrow \begin{bmatrix} \dot{U}_1 \\ \dot{I}_1 \end{bmatrix} = \mathbf{A} \cdot \begin{bmatrix} \dot{U}_2 \\ \dot{I}_2 \end{bmatrix}$$

Chapter Outline

8. Two-Port Networks
   - Two-Port Network Models
   - Methods for Determining Two-Port Network Parameters
   - Methods for Solving Circuits with Two-Port Networks
   - Thévenin–Norton for Circuits Containing Two-Port Networks
   - Transfer Function Concept in Electric Circuits
   - Interconnection of Two-Port Networks

Methods for Determining Two-Port Network Parameters

- The parameter matrices of a two-port network depend only on the network's own circuit structure, not on external quantities

Two common methods for determining two-port network parameters

1. Measure the open-circuit or short-circuit state of one port of the network
2. Given one matrix of the network, determine the remaining matrices

Open/Short-Circuit Measurement Method

Matrix **A**

$$\begin{cases} \dot{U}_1 = a_{11} \cdot \dot{U}_2 + a_{12} \cdot \dot{I}_2 \\ \dot{I}_1 = a_{21} \cdot \dot{U}_2 + a_{22} \cdot \dot{I}_2 \end{cases}$$

- Open-circuit port 2
  $\Rightarrow \dot{I}_2 = 0$

$$a_{11} = \left.\frac{\dot{U}_1}{\dot{U}_2}\right|_{\dot{I}_2=0}$$

$$a_{21} = \left.\frac{\dot{I}_1}{\dot{U}_2}\right|_{\dot{I}_2=0}$$

- Short-circuit port 2
  $\Rightarrow \dot{U}_2 = 0$

$$a_{12} = \left.\frac{\dot{U}_1}{\dot{I}_2}\right|_{\dot{U}_2=0}$$

$$a_{22} = \left.\frac{\dot{I}_1}{\dot{I}_2}\right|_{\dot{U}_2=0}$$

- a_{11} and a_{22} are dimensionless
- a_{12} has unit Ω; a_{21} has unit S

Open/Short-Circuit Measurement Method

Matrix **Z**

$$\begin{cases} \dot{U}_1 = z_{11} \cdot \dot{I}_1 + z_{12} \cdot \dot{I}_2 \\ \dot{U}_2 = z_{21} \cdot \dot{I}_1 + z_{22} \cdot \dot{I}_2 \end{cases}$$

- Open-circuit port 2
  $\Rightarrow \dot{I}_2 = 0$

$$z_{11} = \left.\frac{\dot{U}_1}{\dot{I}_1}\right|_{\dot{I}_2=0}$$

$$z_{21} = \left.\frac{\dot{U}_2}{\dot{I}_1}\right|_{\dot{I}_2=0}$$

- Open-circuit port 1
  $\Rightarrow \dot{I}_1 = 0$

$$z_{12} = \left.\frac{\dot{U}_1}{\dot{I}_2}\right|_{\dot{I}_1=0}$$

$$z_{22} = \left.\frac{\dot{U}_2}{\dot{I}_2}\right|_{\dot{I}_1=0}$$

- All parameters of matrix **Z** have unit Ω

Open/Short-Circuit Measurement Method

Matrix $\mathbf{Y}$

$$\begin{cases} \dot{I}_1 = y_{11} \cdot \dot{U}_1 + y_{12} \cdot \dot{U}_2 \\ \dot{I}_2 = y_{21} \cdot \dot{U}_1 + y_{22} \cdot \dot{U}_2 \end{cases}$$

- Short-circuit port 2 $\Rightarrow \dot{U}_2 = 0$

$$y_{11} = \left.\frac{\dot{I}_1}{\dot{U}_1}\right|_{\dot{U}_2=0}$$

$$y_{21} = \left.\frac{\dot{I}_2}{\dot{U}_1}\right|_{\dot{U}_2=0}$$

- Short-circuit port 1 $\Rightarrow \dot{U}_1 = 0$

$$y_{12} = \left.\frac{\dot{I}_1}{\dot{U}_2}\right|_{\dot{U}_1=0}$$

$$y_{22} = \left.\frac{\dot{I}_2}{\dot{U}_2}\right|_{\dot{U}_1=0}$$

- All parameters of matrix $\mathbf{Y}$ have unit S

Matrix Conversion Method

- The structure of a two-port network is uniquely determined

⇒ The parameter matrices can be equivalently converted into one another

- It is easy to see:
  - $\mathbf{Y} = \mathbf{Z}^{-1}$
  - $\mathbf{H} = \mathbf{G}^{-1}$
- For other matrices, conversion formulas can be derived
- Example: from **A** to **Z**

$$\mathbf{Z} = \begin{bmatrix} \dfrac{a_{11}}{a_{21}} & -\dfrac{\det(\mathbf{A})}{a_{21}} \\ \dfrac{1}{a_{21}} & -\dfrac{a_{22}}{a_{21}} \end{bmatrix}$$

- For other conversions: consult tables in the references

⚠ Note that the reference direction convention for current i_2 in different materials affects the signs of the parameters

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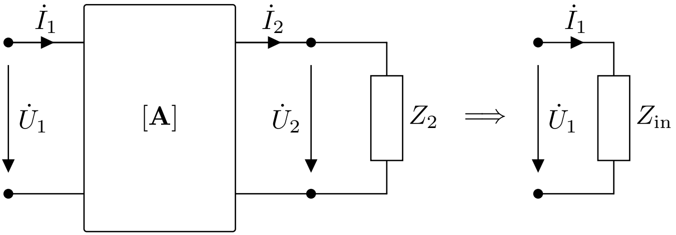
Equivalent-Impedance Method with Matrix **A**

- Suitable for circuits with a source at the input port and a load at the output port
- Procedure:
  - Find the equivalent impedance looking into the input port
  - Use the relation based on **A** → convenient for equivalent impedance

  ⇒ determine the current-voltage pair at the input port

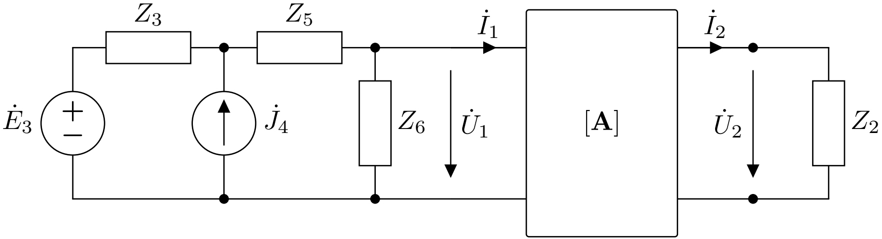
  ⇒ determine the current-voltage pair at the output port



$$Z_{\text{in}} = \frac{\dot{U}_1}{\dot{I}_1} = \frac{a_{11} \cdot \dot{U}_2 + a_{12} \cdot \dot{I}_2}{a_{21} \cdot \dot{U}_2 + a_{22} \cdot \dot{I}_2} \text{ with } \dot{U}_2 = Z_2 \cdot \dot{I}_2 \Rightarrow Z_{\text{in}} = \frac{a_{11} \cdot Z_2 + a_{12}}{a_{21} \cdot Z_2 + a_{22}}$$

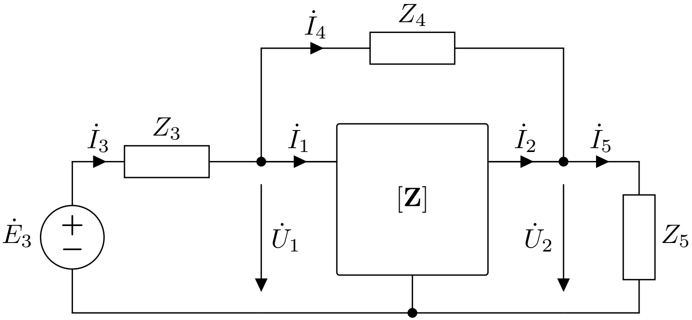
Equivalent-Impedance Method with Matrix **A**

Example 8.1: Determine the current-voltage pairs at the 2 ports



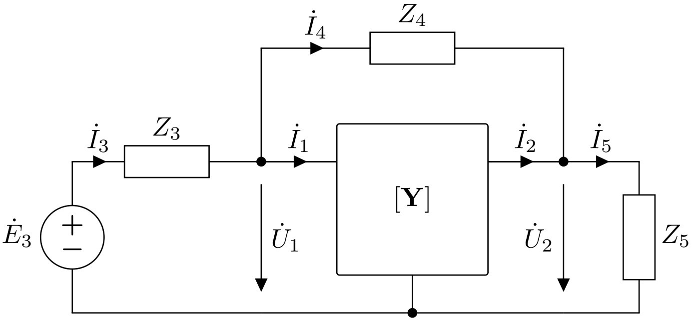
Loop Analysis with Matrix **Z**

Example 8.2: Determine the current-voltage pairs at the 2 ports



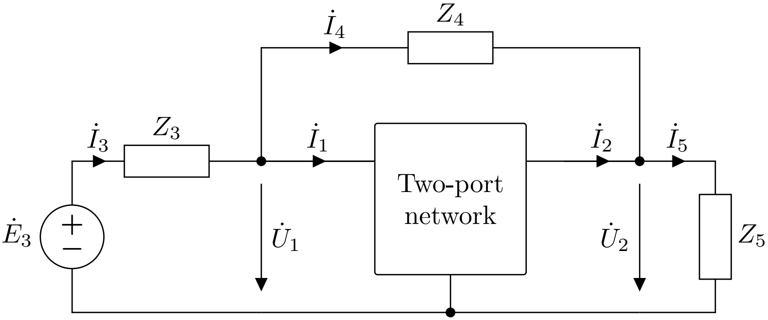
Nodal Analysis with Matrix $\mathbf{Y}$

Example 8.3: Determine the current-voltage pairs at the 2 ports



T and Π Equivalent Two-Port Networks

Equivalent model of a two-port network



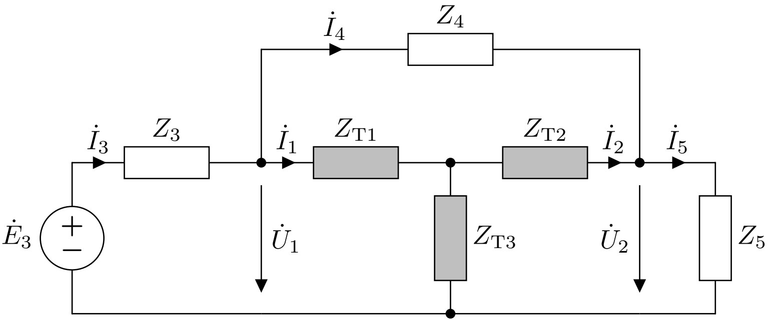
- Two-port networks with the same parameter set are equivalent

$\Rightarrow$ Equivalent circuit models of two-port networks can be constructed

- Two common models are the T network and the Π network

T and Π Equivalent Two-Port Networks

T-model of a two-port network

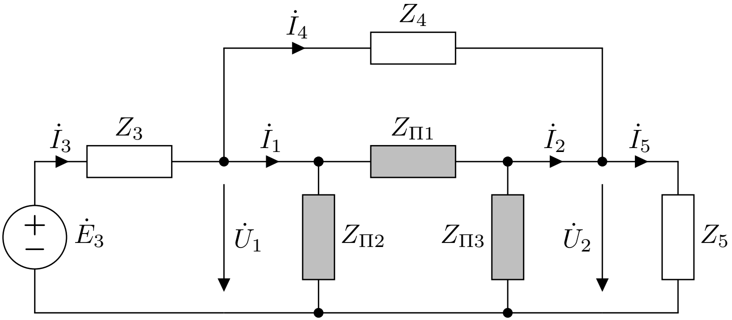


- Determine characteristic matrices from the equivalent impedance set: based on the short-circuit and open-circuit of each port

⇒ From there, formulas for equivalent impedances can be derived from the characteristic matrices.

T and Π Equivalent Two-Port Networks

Π-model of a two-port network



- Similar to the T network
- Other equivalent diagrams also exist

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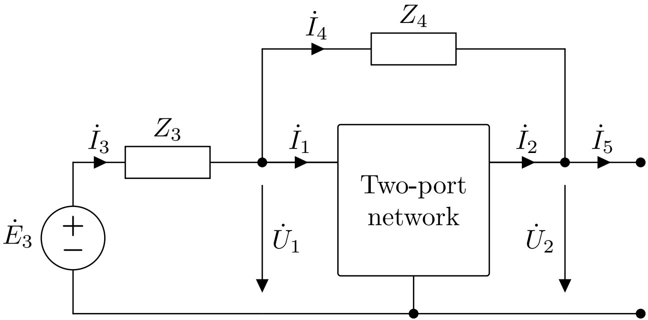
One-Port Network Containing a Two-Port Network

- A linear circuit containing a two-port network is also linear

⇒ The Thévenin–Norton theorems still hold

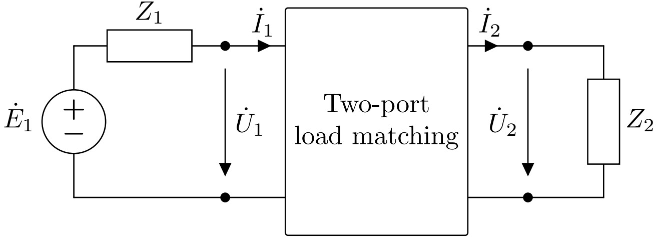
Example 8.4: Find the Thévenin–Norton equivalent diagram for network [**Y**]



⚠ Note: current $\dot{I}_2$ under this open-circuit condition $\neq 0$

Using a Two-Port Network for Impedance Matching

- A two-port network can be connected between a source and a load to achieve maximum power transfer
- Procedure: Determine the equivalent impedance, then match the source and load impedances



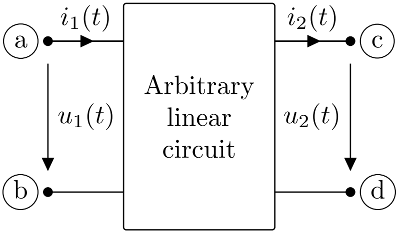
Chapter Outline

Transfer Function Concept in Electric Circuits

- So far, characteristic equations have been constructed as relations between current and voltage on elements
- These relations can be extended to current and voltage state variables at arbitrary positions in the circuit
- For a linear circuit, these relations are definite and unique

$\Rightarrow$ They are called **linear relations** between state variables

$\Rightarrow$ The ratio between state variables is called a **transfer function**

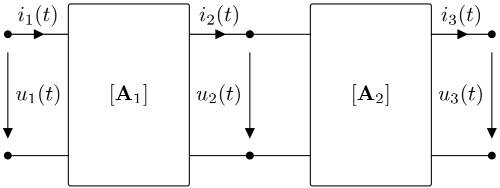


- Voltage transfer function $k_u = \dfrac{u_2}{u_1}$
- Current transfer function: $k_i = \dfrac{i_2}{i_1}$
- Impedance transfer function $k_Z = \dfrac{u_2}{i_1}$ and admittance transfer function $k_Y = \dfrac{i_2}{u_1}$

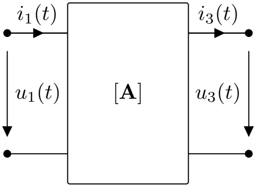
Chapter Outline

Cascaded Two-Port Networks

For cascade connection, matrix $\mathbf{A}$ is the most convenient



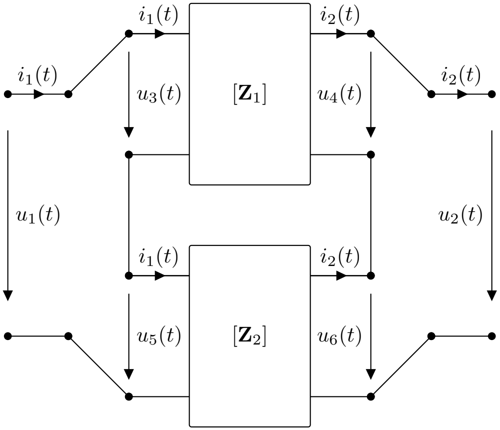
$$\Rightarrow \mathbf{A} = \mathbf{A}_1 \cdot \mathbf{A}_2$$



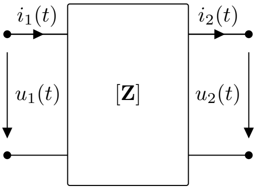
- They share the current and voltage at the connected port

Series-Connected Two-Port Networks

For series connection, matrix $\mathbf{Z}$ is the most convenient



$$\Rightarrow \mathbf{Z} = \mathbf{Z}_1 + \mathbf{Z}_2$$



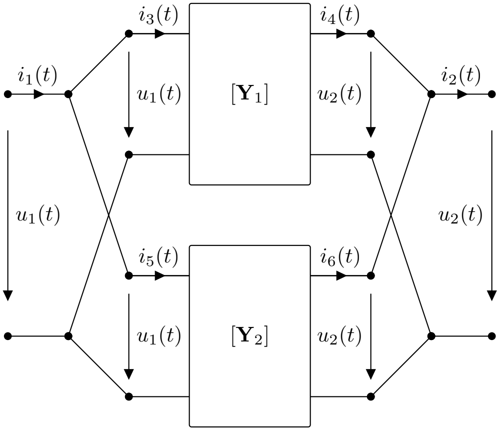
At each port:

- Common current
- Voltage equals the algebraic sum of voltages

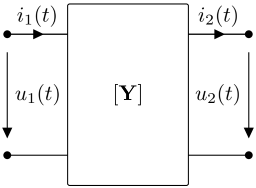
$\Rightarrow$ This is called series connection

Parallel-Connected Two-Port Networks

For parallel connection, matrix $\mathbf{Y}$ is the most convenient



$$\Rightarrow \mathbf{Y} = \mathbf{Y}_1 + \mathbf{Y}_2$$



At each port:

- Common voltage
- Current equals the algebraic sum of currents

$\Rightarrow$ This is called parallel connection

Course Outline

Chapter Outline

9 Circuits with Multiple Frequencies and Non-Sinusoidal Sources
- Applying the Superposition Principle in Multi-Frequency Circuits
- Power in Multi-Frequency Circuits
- Circuits with Non-Sinusoidal Sources – Fourier Method
- Frequency Response of Transfer Functions – Electric Filters

Chapter Outline

Circuits with Multiple Frequencies

Circuits with multiple frequencies

- So far, we have considered DC circuits (which can be regarded as $f = 0$) and AC circuits whose sources have the same frequency
- In many cases, sources in a circuit may have different frequencies

Problem

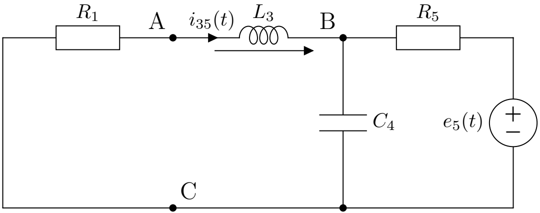
- The phasor method can only be used for AC circuits whose sources have the same frequency

Solution

- Solve the circuit for each frequency in turn by suppressing sources with other frequencies (short-circuit voltage sources, open-circuit current sources)
- Superpose the solutions *in the time domain*

Circuits with Multiple Frequencies

Example 9.1: Determine current $i_3(t)$



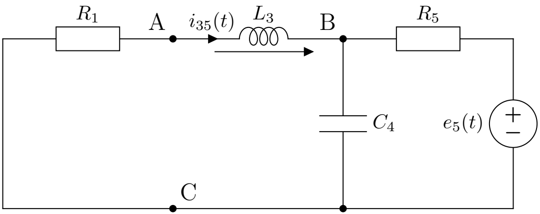
$E_1 = 15\,\text{V}$; $R_1 = 10\,\Omega$;
$j_2(t) = \sqrt{2}\sin(5t + 15^\circ)\,\text{A}$;
$L_3 = 1\,\text{H}$; $C_4 = 0{,}02\,\text{F}$;
$R_5 = 5\,\Omega$;
$e_5(t) = 12\sqrt{2}\sin(10t + 30^\circ)\,\text{V}$

Solution

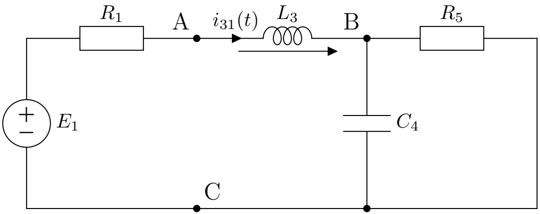
- $i_3(t) = i_{31}(t) + i_{32}(t) + i_{35}(t)$
- $i_{31}(t)$ is the current through L_3 when only E_1 on, $j_2(t)$ and $e_5(t)$ off
- $i_{32}(t)$ is the current through L_3 when only $j_2(t)$ on, E_1 and $e_5(t)$ off
- $i_{35}(t)$ is the current through L_3 when only $e_5(t)$ on, $j_2(t)$ and E_1 off

Circuits with Multiple Frequencies

Example 9.1: Determine current $i_3(t)$



$$E_1 = 15\,\text{V};\; R_1 = 10\,\Omega;$$
$$j_2(t) = \sqrt{2}\sin\left(5t + 15^\circ\right)\text{A};$$
$$L_3 = 1\,\text{H};\; C_4 = 0{,}02\,\text{F};$$
$$R_5 = 5\,\Omega;$$
$$e_5(t) = 12\sqrt{2}\sin\left(10t + 30^\circ\right)\text{V}$$

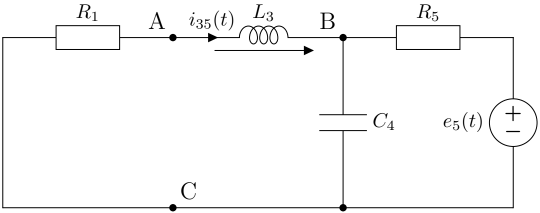


Component $i_{31}(t) = I_{31}$

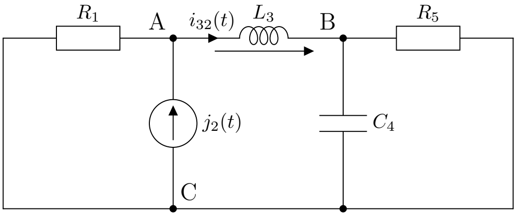
$$I_{31} = \frac{E_1}{R_1 + R_5} = 1\,\text{A}$$

Circuits with Multiple Frequencies

Example 9.1: Determine current $i_3(t)$



$E_1 = 15\,\mathrm{V};\ R_1 = 10\,\Omega;$
$j_2(t) = \sqrt{2}\sin(5t + 15^\circ)\,\mathrm{A};$
$L_3 = 1\,\mathrm{H};\ C_4 = 0{,}02\,\mathrm{F};$
$R_5 = 5\,\Omega;$
$e_5(t) = 12\sqrt{2}\sin(10t + 30^\circ)\,\mathrm{V}$



Component $i_{32}(t)$

Convert the circuit to the complex domain

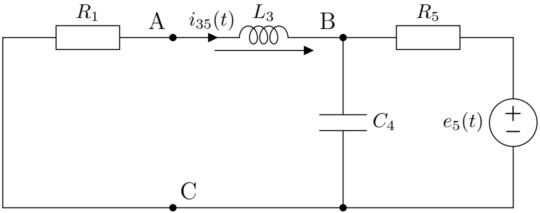
$Z_{L_3} = j\omega_2 L_3 = j5\,\Omega$

$Z_{C_4} = \dfrac{1}{j\omega_2 C_4} = -j10\,\Omega$

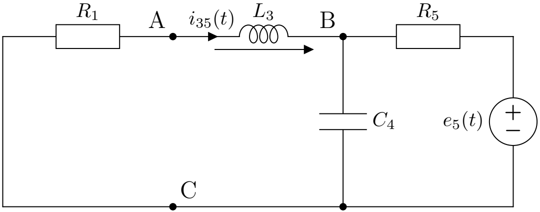
determine $i_{32}(t) = \dots$

Circuits with Multiple Frequencies

Example 9.1: Determine current $i_3(t)$



$E_1 = 15\,\text{V}$; $R_1 = 10\,\Omega$;
$j_2(t) = \sqrt{2}\sin(5t + 15°)$ A;
$L_3 = 1\,\text{H}$; $C_4 = 0{,}02\,\text{F}$;
$R_5 = 5\,\Omega$;
$e_5(t) = 12\sqrt{2}\sin(10t + 30°)$ V



Component $i_{35}(t)$

Convert the circuit to the complex domain

$Z_{L_3} = j\omega_5 L_3 = j10\,\Omega$

$Z_{C_4} = \dfrac{1}{j\omega_5 C_4} = -j5\,\Omega$

determine $i_{35}(t) = \ldots$

Chapter Outline

9 Circuits with Multiple Frequencies and Non-Sinusoidal Sources

- Applying the Superposition Principle in Multi-Frequency Circuits
- Power in Multi-Frequency Circuits
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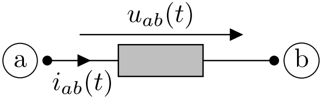
Review: Instantaneous Power and Average Power



- Instantaneous absorbed power:

$$p(t) = u_{ab}(t) \cdot i_{ab}(t)$$

- Instantaneous supplied power:

$$p_{\text{sup}}(t) = -p(t) = u_{ba}(t) \cdot i_{ab}(t) = u_{ab}(t) \cdot i_{ba}(t)$$

- Average power is the average value of instantaneous power

$$P = \frac{1}{T} \int_0^T p(t) \cdot dt$$

Power in Multi-Frequency Circuits

- When current and voltage quantities have multiple frequency components

$$u(t) = U_0 + U_1 \sin(\omega_1 t + \varphi_1) + U_2 \sin(\omega_2 t + \varphi_2) + \dots$$

$$i(t) = I_0 + I_1 \sin(\omega_1 t + \theta_1) + I_2 \sin(\omega_2 t + \theta_2) + \dots$$

- Instantaneous power:

$$p(t) = u(t) \cdot i(t) = U_0 I_0 + \sum_i U_i I_i \sin(\omega_i t + \varphi_i) \sin(\omega_i t + \theta_i)$$
$$+ \sum_{i \neq j} U_i I_j \sin(\omega_i t + \varphi_i) \sin(\omega_j t + \theta_j)$$

Power in Multi-Frequency Circuits

- Instantaneous power:

$$p(t) = u(t) \cdot i(t) = U_0 I_0 + \sum_i U_i I_i \sin(\omega_i t + \varphi_i) \sin(\omega_i t + \theta_i)$$

$$+ \sum_{i \neq j} U_i I_j \sin(\omega_i t + \varphi_i) \sin(\omega_j t + \theta_j)$$

- Average power:

$$P = \frac{1}{T} \int_0^T p(t) dt = U_0 I_0 + \sum_i \frac{1}{T} \int_0^T (U_i I_i \sin(\omega_i t + \varphi_i) \sin(\omega_i t + \theta_i))\, dt$$

$$+ \sum_{i \neq j} \frac{1}{T} \int_0^T (U_i I_j \sin(\omega_i t + \varphi_i) \sin(\omega_j t + \theta_j))\, dt$$

$$= U_0 I_0 + \sum_i \frac{U_i I_i}{2} \cos(\varphi_i - \theta_i) + 0 = P_{\mathrm{DC}} + \sum_i P_{\omega_i}$$

RMS Value in Multi-Frequency Circuits

- $u(t) = U_0 \to U_{\text{rms}} = U_0$
- $u(t) = U_1 \sin(\omega_1 t + \varphi_1) \to U_{\text{rms}} = \dfrac{U_1}{\sqrt{2}}$
- $u(t) = U_0 + U_1 \sin(\omega_1 t + \varphi_1) + U_2 \sin(\omega_2 t + \varphi_2) + \dots$

$$\to U_{\text{rms}} = \sqrt{\left(U_0^2 + \frac{U_1^2}{2} + \frac{U_2^2}{2} + \dots\right)}$$

⚠ Note: in multi-frequency circuits, average power must not be determined from RMS current and voltage values

$$P_{\text{avg}} \neq U_{\text{rms}} \cdot I_{\text{rms}}$$

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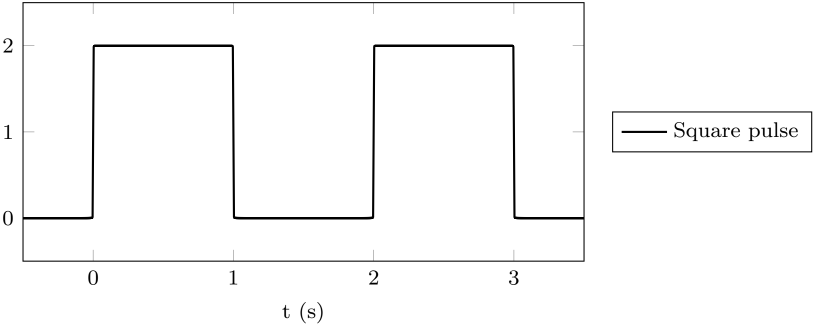
Circuits with Non-Sinusoidal Sources

Example 9.2: Fourier expansion of a square pulse

- Given a square voltage pulse with amplitude 1 V, period 1 s, and offset 1 V

$$u(t) = 1 + \frac{1}{4\pi}\sum_{n=1}^{\infty}\frac{\sin\left(n\pi t\right)}{n} \text{ for odd } n$$

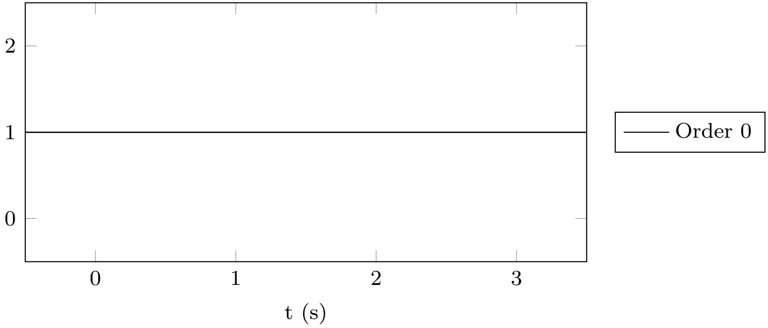


Circuits with Non-Sinusoidal Sources

Example 9.2: Fourier expansion of a square pulse

- DC component (order 0)

$$u(t) = 1$$

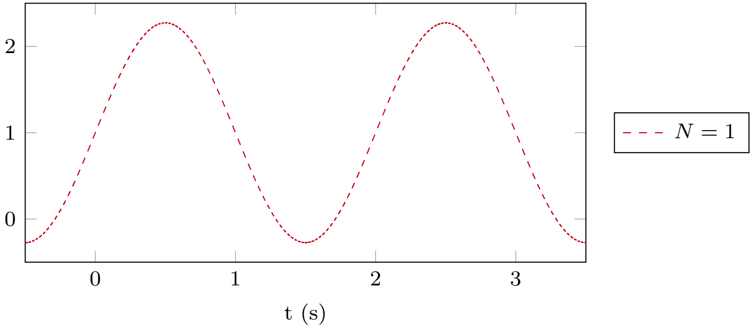


Circuits with Non-Sinusoidal Sources

Example 9.2: Fourier expansion of a square pulse

- Expansion up to the 1st harmonic component

$$u(t) = 1 + \frac{1}{4\pi}\sin(\pi t)$$

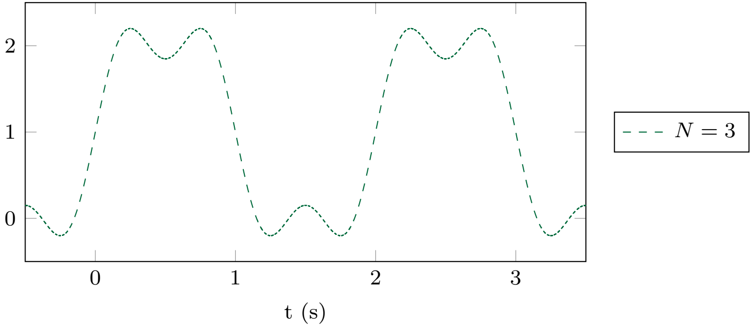


Circuits with Non-Sinusoidal Sources

Example 9.2: Fourier expansion of a square pulse

- Expansion up to the 3rd harmonic component

$$u(t) = 1 + \frac{1}{4\pi}\sin(\pi t) + \frac{1}{4\pi}\frac{\sin(3\pi t)}{3}$$

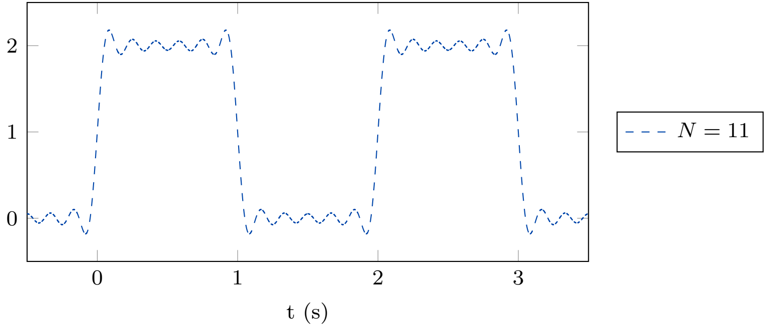


Circuits with Non-Sinusoidal Sources

Example 9.2: Fourier expansion of a square pulse

- Expansion up to the 11th harmonic component

$$u(t) = 1 + \frac{1}{4\pi}\sum_{n=1}^{11}\frac{\sin\left(n\pi t\right)}{n} \text{ for odd } n$$

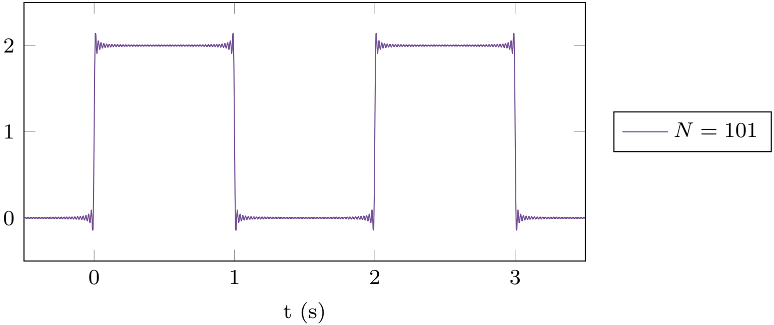


Circuits with Non-Sinusoidal Sources

Example 9.2: Fourier expansion of a square pulse

- Expansion up to the 101st harmonic component

$$u(t) = 1 + \frac{1}{4\pi}\sum_{n=1}^{101}\frac{\sin(n\pi t)}{n} \text{ for odd } n$$

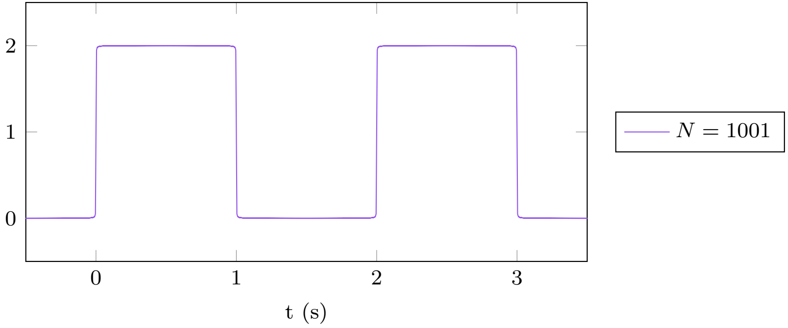


Circuits with Non-Sinusoidal Sources

Example 9.2: Fourier expansion of a square pulse

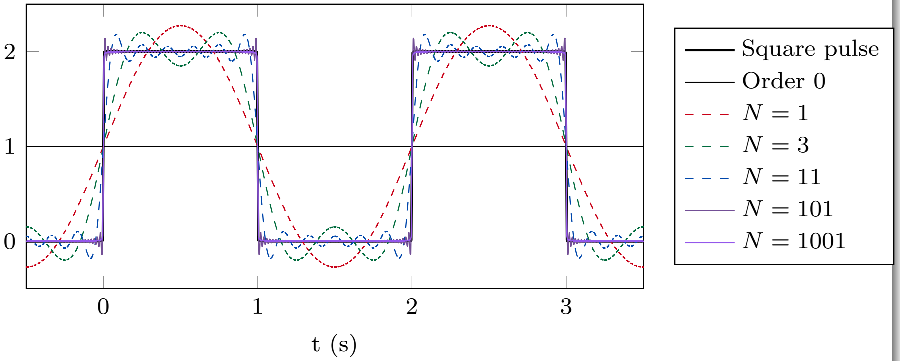
- Expansion up to the 1001st harmonic component

$$u(t) = 1 + \frac{1}{4\pi}\sum_{n=1}^{1001}\frac{\sin(n\pi t)}{n} \text{ for odd } n$$



Circuits with Non-Sinusoidal Sources

Example 9.2: Fourier expansion of a square pulse



- Exercise: Determine the expansions of other periodic signals such as triangular waves, sawtooth waves, etc.

Chapter Outline

9 Circuits with Multiple Frequencies and Non-Sinusoidal Sources
- Applying the Superposition Principle in Multi-Frequency Circuits
- Power in Multi-Frequency Circuits
- Circuits with Non-Sinusoidal Sources – Fourier Method
- Frequency Response of Transfer Functions – Electric Filters

Frequency Response of Transfer Functions

- A transfer function is a proportional relation between current-voltage state variables in a circuit

⇒ The transfer function is formed from the circuit impedances

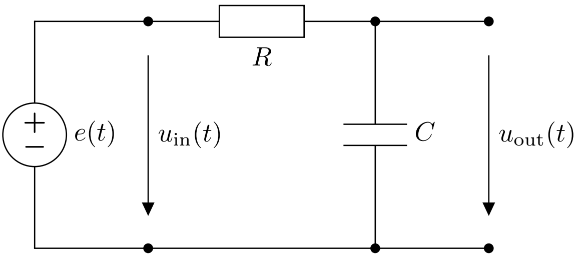
- Capacitors and inductors have different complex impedances at different frequencies in the circuit

⇒ The transfer function is a complex function whose magnitude and phase characteristics depend on frequency

⇒ The concept of **frequency–magnitude–phase response** of a transfer function

Frequency Response of Transfer Functions

Example 9.3: Consider the voltage transfer function of an RC circuit

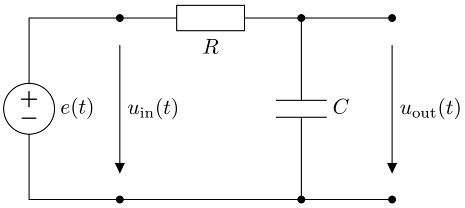


$$H(j\omega) = \frac{\dot{U}_{\text{out}}}{\dot{U}_{\text{in}}} = \frac{Z_C}{R + Z_C} = \frac{\dfrac{1}{j\omega C}}{R + \dfrac{1}{j\omega C}}$$

$$= \frac{1}{j\omega RC + 1}$$

Frequency Response of Transfer Functions

Example 9.3: Consider the voltage transfer function of an RC circuit



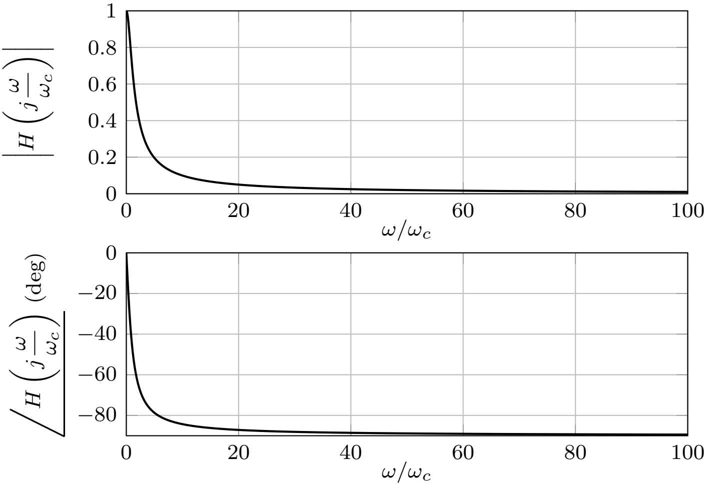
- Set $RC = \dfrac{1}{\omega_c}$
- ω_c is called the *cutoff frequency*

$\Rightarrow H(j\omega) = \dfrac{1}{j\dfrac{\omega}{\omega_c} + 1}$

- Magnitude response $|H(j\omega)| = \dfrac{1}{\sqrt{\left(\dfrac{\omega}{\omega_c}\right)^2 + 1}}$
- Phase response $\angle H(j\omega) = \angle 1 - \angle\left(j\dfrac{\omega}{\omega_c} + 1\right) = -\arctan\left(\dfrac{\omega}{\omega_c}\right)$

Frequency Response of Transfer Functions

Both the plot and the formulas are inconvenient



Frequency Response of Transfer Functions

Decibel magnitude (dB) and logarithmic frequency scale

- The log function is convenient because it converts multiplication and division into addition and subtraction $\rightarrow$ operations can be performed directly on the plot
- A logarithmic scale makes frequency plots easier to read than a linear scale

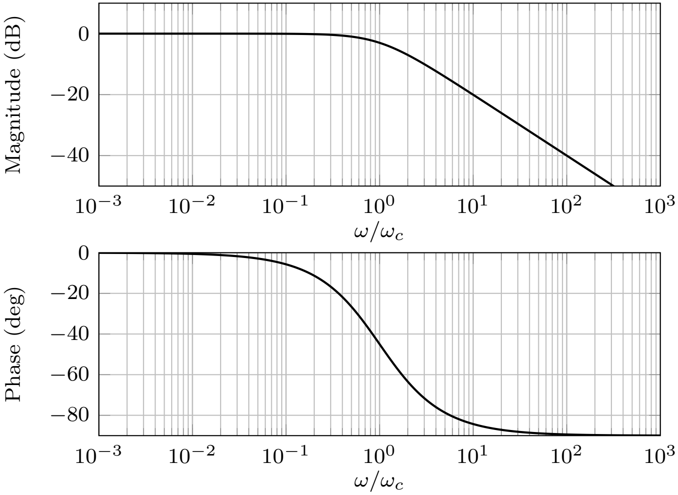
$\Rightarrow$ The dB magnitude of the magnitude response is defined as $20\log_{10}(|H(j\omega)|)^*$

$\Rightarrow$ A plot in dB and on a log scale is called a **Bode plot**

The choice of unit and formula is historical. Initially, the bel was named to recognize telephone inventor Alexander Graham Bell. This unit is the base-10 logarithm of the ratio between two powers: $\log_{10}(P_2/P_1)$ bels $= 10\log_{10}(P_2/P_1)$ decibels. Meanwhile, signal power is proportional to the square of signal amplitude $\rightarrow 10\log_{10}(P_2/P_1) = 10\log_{10}(|H(j\omega)|^2) = 20\log_{10}(|H(j\omega)|)$ dB. (Lathi, Linear Systems and Signals)

Frequency Response of Transfer Functions

Bode plot of a first-order RC low-pass filter



Course Outline

Chapter Outline

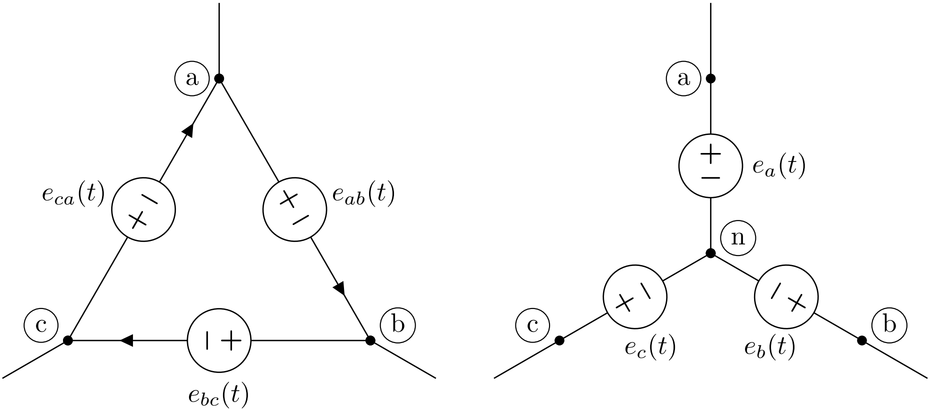
10. Three-Phase Circuits
    - Three-Phase Circuit Concept
    - Balanced Three-Phase Circuits
    - Unbalanced Three-Phase Circuits

Chapter Outline

10 Three-Phase Circuits
- Three-Phase Circuit Concept
- Balanced Three-Phase Circuits
- Unbalanced Three-Phase Circuits

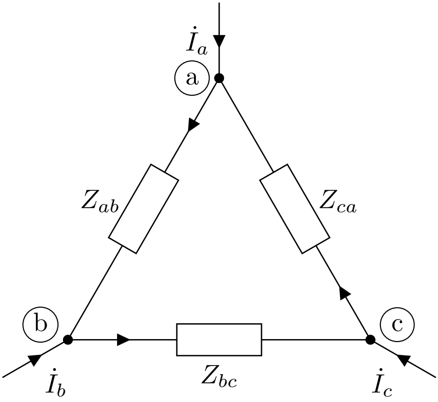
Three-Phase Circuit Concept

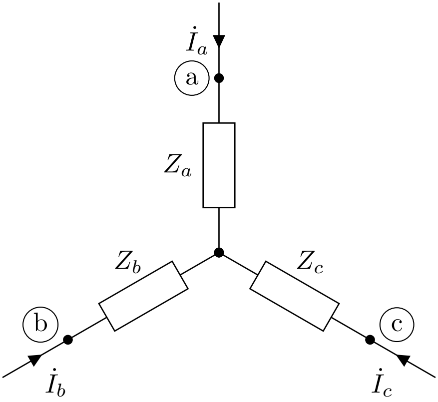
- A three-phase source system is a set of 3 AC voltage sources with phase shifts
- A three-phase source set can be connected in wye (Y) or delta (Δ)



Three-Phase Circuit Concept

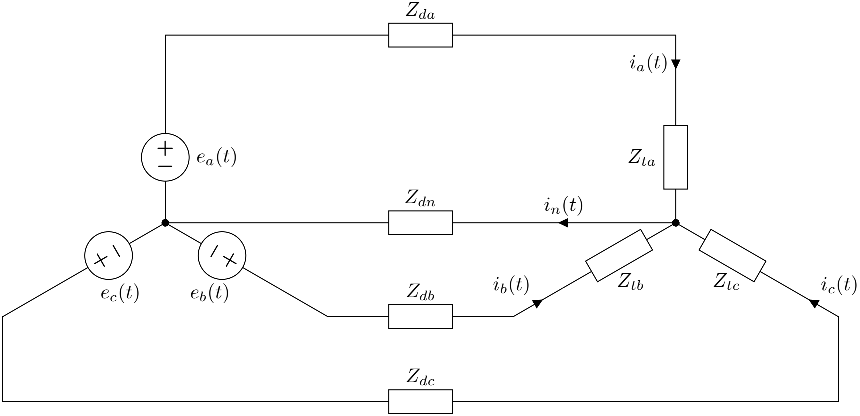
- A three-phase load can also be connected in wye (Y) or delta (Δ)





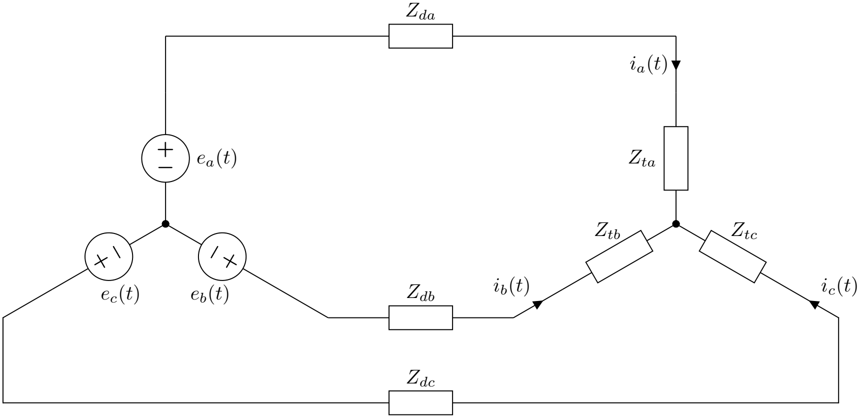
Three-Phase Circuit Concept

- When connecting source and load, line impedance can be considered
- A neutral wire can be connected



Three-Phase Circuit Concept

- Or the neutral wire can be omitted



- And other three-phase circuit connections

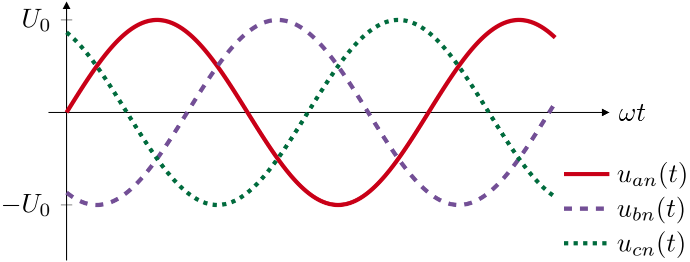
Chapter Outline

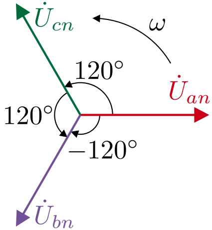
Balanced Three-Phase Circuits

- Balanced load: $Z_{ta} = Z_{tb} = Z_{tc}$
- Balanced line: $Z_{da} = Z_{db} = Z_{dc}$
- Balanced source: equal amplitudes and phase shifts of 120°
- Let $a = 1\angle{-120°}$ be the $-120°$ rotation operator

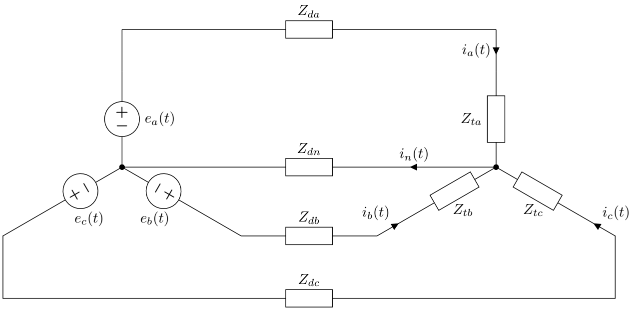
$\Rightarrow$ $\dot{U}_{bn} = a \cdot \dot{U}_{an}$; $\dot{U}_{cn} = a \cdot \dot{U}_{bn} = a^2 \cdot \dot{U}_{an}$; $\dot{U}_{an} = a \cdot \dot{U}_{cn}$

$\Rightarrow$ $\dot{U}_{an} + \dot{U}_{bn} + \dot{U}_{cn} = \left(1 + a + a^2\right) \cdot \dot{U}_{an} = 0$





Solving Three-Phase Circuits



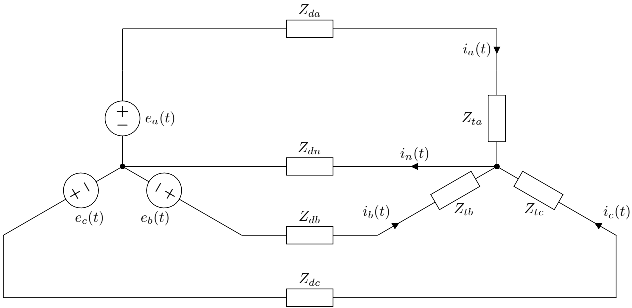
- Case with a neutral wire → Nodal Analysis

$$\dot{V}\left(\frac{1}{Z_{ta}+Z_{da}}+\frac{1}{Z_{tb}+Z_{db}}+\frac{1}{Z_{tc}+Z_{dc}}+\frac{1}{Z_{dn}}\right)$$
$$=\frac{\dot{E}_a}{Z_{ta}+Z_{da}}+\frac{\dot{E}_b}{Z_{tb}+Z_{db}}+\frac{\dot{E}_c}{Z_{tc}+Z_{dc}}$$

Solving Three-Phase Circuits

Define phase impedances

$$Z_a = Z_{ta} + Z_{da}$$
$$Z_b = Z_{tb} + Z_{db}$$
$$Z_c = Z_{tc} + Z_{dc}$$



⇒ Node potential

$$\dot{V} = \frac{\dfrac{\dot{E}_a}{Z_a} + \dfrac{\dot{E}_b}{Z_b} + \dfrac{\dot{E}_c}{Z_c}}{\dfrac{1}{Z_a} + \dfrac{1}{Z_b} + \dfrac{1}{Z_c} + \dfrac{1}{Z_n}}$$

Solving Balanced Three-Phase Circuits

- Balanced three-phase circuit

  $Z_a = Z_b = Z_c$

  $\dot{E}_a + \dot{E}_b + \dot{E}_c = 0$

$$\Rightarrow \dot{V} = \frac{\dfrac{\dot{E}_a}{Z_a} + \dfrac{\dot{E}_b}{Z_b} + \dfrac{\dot{E}_c}{Z_c}}{\dfrac{1}{Z_a} + \dfrac{1}{Z_b} + \dfrac{1}{Z_c} + \dfrac{1}{Z_n}} = 0$$

$$\Rightarrow \dot{I}_a = \frac{\dot{E}_a}{Z_a};\ \dot{I}_b = \frac{\dot{E}_b}{Z_b};\ \dot{I}_c = \frac{\dot{E}_c}{Z_c};\ \dot{I}_n = 0$$

$\Rightarrow$ Balanced three-phase currents

$$\dot{I}_b = a \cdot \dot{I}_a;\ \dot{I}_c = a \cdot \dot{I}_b = a^2 \cdot \dot{I}_a;\ \dot{I}_a = a \cdot \dot{I}_c$$

$\Rightarrow$ Balanced three-phase load voltages

$$\dot{U}_{tb} = a \cdot \dot{U}_{ta};\ \dot{U}_{tc} = a \cdot \dot{U}_{tb} = a^2 \cdot \dot{U}_{ta};\ \dot{U}_{ta} = a \cdot \dot{U}_{tc}$$

with $a = 1\underline{/-120^\circ}$

Power in Balanced Three-Phase Circuits

- Load power

$$\begin{aligned} P_{ta} &= |\dot{U}_{ta}| \cdot |\dot{I}_{ta}| \cdot \cos\left(\underline{/\dot{U}_{ta}} - \underline{/\dot{I}_{ta}}\right) \\ &= |\dot{U}_{tb}| \cdot |\dot{I}_{tb}| \cdot \cos\left(\left(\underline{/\dot{U}_{tb}} + 120^\circ\right) - \left(\underline{/\dot{I}_{tb}} + 120^\circ\right)\right) \\ &= |\dot{U}_{tb}| \cdot |\dot{I}_{tb}| \cdot \cos\left(\underline{/\dot{U}_{tb}} - \underline{/\dot{I}_{tb}}\right) \\ &= P_{tb} = P_{tc} \end{aligned}$$

$\Rightarrow$ The absorbed power of the load on each phase is equal

$\Rightarrow$ Similarly, the supplied power of the source on each phase is equal

- The total instantaneous power at every instant is constant

$$P = 3 \cdot U_{\text{rms}} \cdot I_{\text{rms}} \cdot \cos(\varphi - \theta) = \frac{3}{2} \cdot U_0 \cdot I_0 \cdot \cos(\varphi - \theta)$$

where φ and θ are the angles of voltage and current on each phase

Chapter Outline

Unbalanced Three-Phase Circuits

- A three-phase circuit whose source or load is unbalanced in amplitude or phase angle
- Solution methods
  - Direct solution
  - Symmetrical-component method
- Extend the concept of a balanced three-phase system: complex coefficient k satisfies

  $\dot{E}_b = k \cdot \dot{E}_a$; $\dot{E}_c = k \cdot \dot{E}_b$; $\dot{E}_a = k \cdot \dot{E}_c$

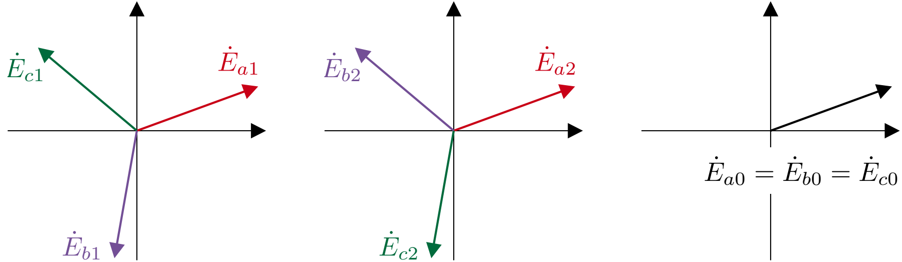
$\Rightarrow$ there are 3 solutions: $k_0 = 1$; $k_1 = a$; $k_2 = a^2$

- Corresponding to 3 symmetrical three-phase systems
  1. Positive sequence $k_1 = a = 1\underline{/-120^\circ}$
     (clockwise $\rightarrow$ opposite to the conventional positive direction)
  2. Negative sequence $k_2 = a^2 = 1\underline{/120^\circ}$
     (counterclockwise $\rightarrow$ same as the conventional positive direction)
  3. Zero sequence $k_0 = 1$ (no rotation)

Unbalanced Three-Phase Circuits

Symmetrical-component method

- Positive sequence
- Negative sequence
- Zero sequence



- An unbalanced three-phase source system is the superposition of 3 symmetrical source components

$\Rightarrow$ Solve 3 equations in 3 unknowns to find the 3 symmetrical components

Symmetrical-Component Method

- For any unbalanced source set $\dot{E}_a; \dot{E}_b; \dot{E}_c$, there exists a source set of symmetrical components $\dot{E}_{a0}; \dot{E}_{a1}; \dot{E}_{a2}$ such that:

$$\begin{aligned}
\dot{E}_a &= \dot{E}_{a0} + \dot{E}_{a1} + \dot{E}_{a2} && = \dot{E}_{a0} + \dot{E}_{a1} + \dot{E}_{a2} \\
\dot{E}_b &= \dot{E}_{a0} + k_1 \cdot \dot{E}_{a1} + k_2 \cdot \dot{E}_{a2} && = \dot{E}_{b0} + \dot{E}_{b1} + \dot{E}_{b2} \\
\dot{E}_c &= \dot{E}_{a0} + k_1^2 \cdot \dot{E}_{a1} + k_2^2 \cdot \dot{E}_{a2} && = \dot{E}_{c0} + \dot{E}_{c1} + \dot{E}_{c2}
\end{aligned}$$

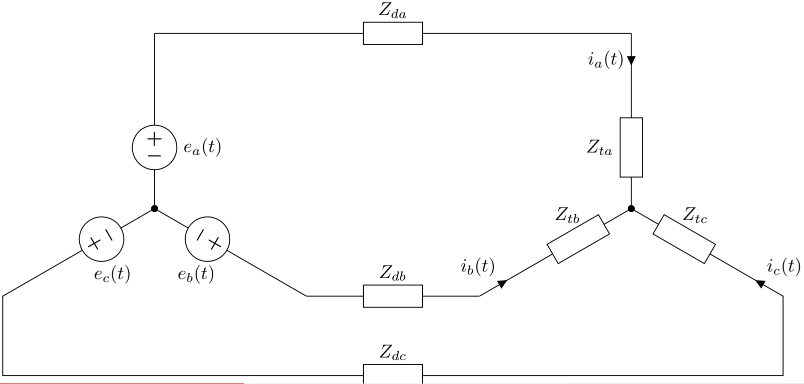
$$\Rightarrow \begin{bmatrix} \dot{E}_a \\ \dot{E}_b \\ \dot{E}_c \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a & a^2 \\ 1 & a^2 & a \end{bmatrix} \cdot \begin{bmatrix} \dot{E}_{a0} \\ \dot{E}_{a1} \\ \dot{E}_{a2} \end{bmatrix} \Rightarrow \begin{bmatrix} \dot{E}_{a0} \\ \dot{E}_{a1} \\ \dot{E}_{a2} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a & a^2 \\ 1 & a^2 & a \end{bmatrix}^{-1} \cdot \begin{bmatrix} \dot{E}_a \\ \dot{E}_b \\ \dot{E}_c \end{bmatrix}$$

$$\begin{aligned}
\dot{E}_{a0} &= \frac{1}{3}\left(\dot{E}_a + \dot{E}_b + \dot{E}_c\right) \\
\Rightarrow \dot{E}_{a1} &= \frac{1}{3}\left(\dot{E}_a + a^2 \cdot \dot{E}_b + a \cdot \dot{E}_c\right) \\
\dot{E}_{a2} &= \frac{1}{3}\left(\dot{E}_a + a \cdot \dot{E}_b + a^2 \cdot \dot{E}_c\right)
\end{aligned}$$

Exercises

BT 10.1

- Given a balanced three-phase circuit $\dot{E}_a = 200\underline{/0^\circ}\,\text{V}$; $Z_{da} + Z_{ta} = 100\underline{/60^\circ}\,\Omega$
- Determine the total power absorbed by the load



Exercises

Exercise 10.1: Determine average power

- Because the circuit is balanced, the current in each phase is:

$$\dot{I}_a = \frac{\dot{E}_a}{Z_{da} + Z_{ta}} = 2\underline{/-60^\circ}\,\text{A}$$

$\Rightarrow$ Average power on one phase

$$P_a = \dot{E}_a \cdot \dot{I}_a \cdot \cos\varphi = 200 \cdot 2 \cdot \cos 60^\circ = 200\,\text{W}$$

$\Rightarrow$ The total average power on the load is 600 W

Exercises

Exercise 10.1: Determine instantaneous power

- Consider a three-phase power system with frequency 50 Hz

$\Rightarrow$ $e_a(t) = 200\sqrt{2}\sin(100\pi t)$ V

$i_a(t) = 2\sqrt{2}\sin(100\pi t - 60^\circ)$ A

$\Rightarrow$ Instantaneous power on phase A

$$
\begin{aligned}
p_a(t) &= [200\sqrt{2}\sin(100\pi t)2\sqrt{2}\sin(100\pi t - 60^\circ)] \\
&= 400[\cos(60^\circ) - \cos(200\pi t - 60^\circ)] \\
&= 200 - 400\cos(200\pi t - 60^\circ)\ \mathrm{W}
\end{aligned}
$$

$\Rightarrow$ Similarly, instantaneous power on phases B and C

$p_b(t) = 200 - 400\cos(200\pi t - 300^\circ)$ W

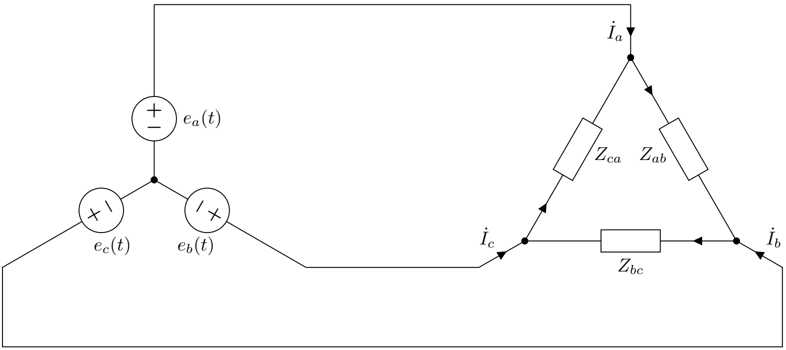
$p_c(t) = 200 - 400\cos(200\pi t - 180^\circ)$ W

- Total instantaneous power $p(t) = p_a(t) + p_b(t) + p_c(t) = 600$ W

Exercises

BT 10.2

- Given a balanced three-phase circuit $\dot{E}_a = 100\underline{/10^\circ}\,\text{V}$; $Z_{ab} = 8 + j4\,\Omega$
- Determine the load phase currents and line currents



Exercises

BT 10.2

- Line voltage: $\dot{U}_{\text{line}} = \dot{U}_{\text{phase}}\sqrt{3}\underline{/30^\circ} = 100\sqrt{3}\underline{/40^\circ}$ V
- Load phase current: $\dot{I}_{pL} = \dfrac{\dot{U}_{\text{line}}}{Z_{ab}} = \dfrac{100\sqrt{3}\underline{/40^\circ}}{8+j4} = 19{,}3655\underline{/13{,}43^\circ}$ A
- Load line current: $\dot{I}_{lL} = \dot{I}_{pL}\sqrt{3}\underline{/-30^\circ} = 19{,}3655\sqrt{3}\underline{/13{,}43^\circ - 30^\circ} = 33{,}542\underline{/-16{,}57^\circ}\,\underline{/A}$
- Use symmetry to infer the other currents

Exercises

BT 10.3

- Given a Y–Y connected three-phase circuit with source set $\dot{E}_a = 220\underline{/10^\circ}\,\text{V}$; $\dot{E}_b = 210\underline{/-100^\circ}\,\text{V}$; $\dot{E}_c = 215\underline{/110^\circ}\,\text{V}$
- and load $Z_a = Z_b = Z_c = 10 + j5\,\Omega$
- Determine the current in each phase

BT 10.4

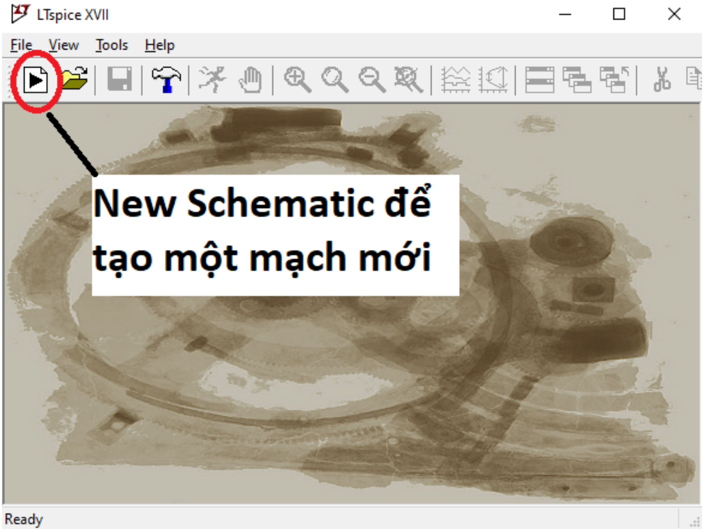
- Given a Y–Y connected three-phase circuit with source set $\dot{E}_a = 220\underline{/10^\circ}\,\text{V}$; $\dot{E}_b = 210\underline{/-100^\circ}\,\text{V}$; $\dot{E}_c = 215\underline{/110^\circ}\,\text{V}$
- and load $Z_a = 10 + j5\,\Omega$; $Z_b = 12 + j4\,\Omega$; $Z_c = 15 + j6\,\Omega$
- Determine the current in each phase

Appendix: Circuit Simulation with LTspice

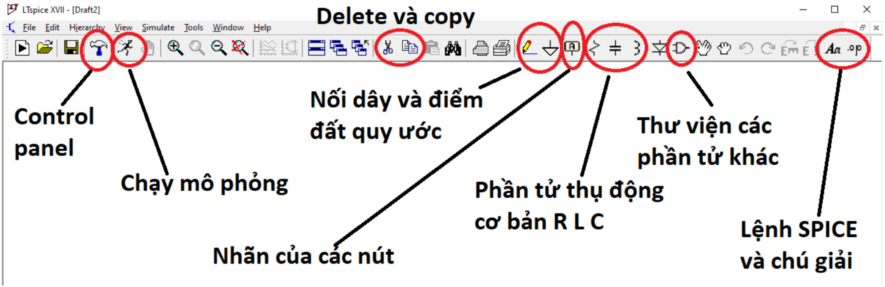
Introduction to SPICE and LTspice

- SPICE is a popular electrical and electronic circuit simulation tool in academia and industry
- SPICE stands for **S**imulation **P**rogram with **I**ntegrated **C**ircuit **E**mphasis, created at the University of California, Berkeley, USA, in the 1970s
- SPICE was originally a tool for describing and solving circuits using command-code descriptions (netlists)
- Later, software tools were developed to help create netlists by drawing circuits through graphical interfaces
- Commercial SPICE-based circuit simulation software: OrCAD PSpice, PrimeSim HSPICE, etc.
- Free SPICE-based circuit simulation software: LTspice, TINA, etc.
- LTspice is a free tool from Linear Technology (now part of Analog Devices) powerful enough for most study and research tasks related to circuits

LTspice Startup Interface



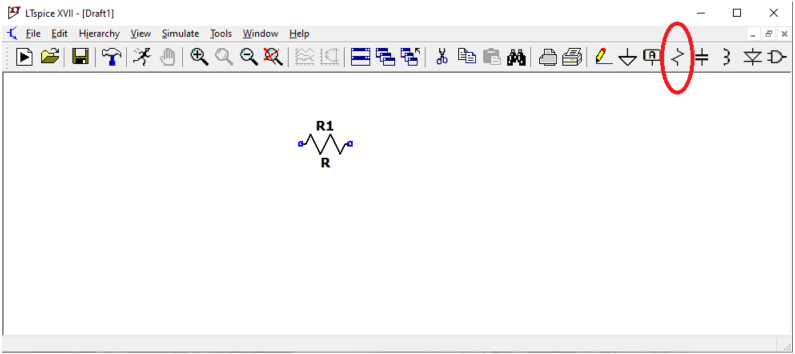
Main Toolbar



Some LTspice operations

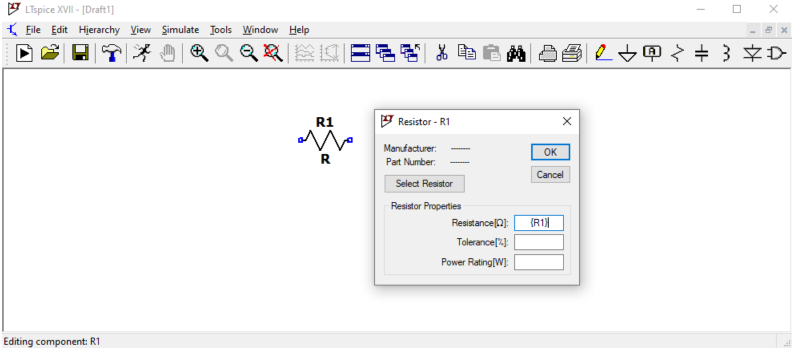
- End an operation: right-click
- Start an operation: choose a tool icon (e.g., scissors mean delete, the hand means move, etc.) and then select the element to act on. (LTspice does not operate in the usual way of selecting an element first and then choosing an operation; but in the opposite order.)

Example 1: Getting Started with LTspice



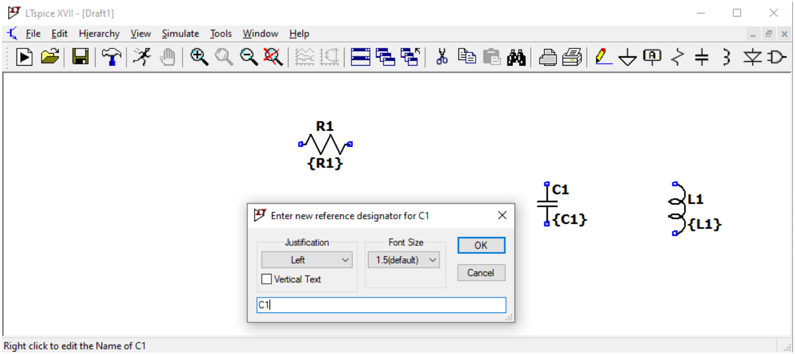
1. Click the button to place a resistor element
2. Before placing the element, press `Ctrl+R` to rotate it horizontally
3. Left-click to place the element
4. Right-click to exit resistor-placement mode

Example 1: Getting Started with LTspice



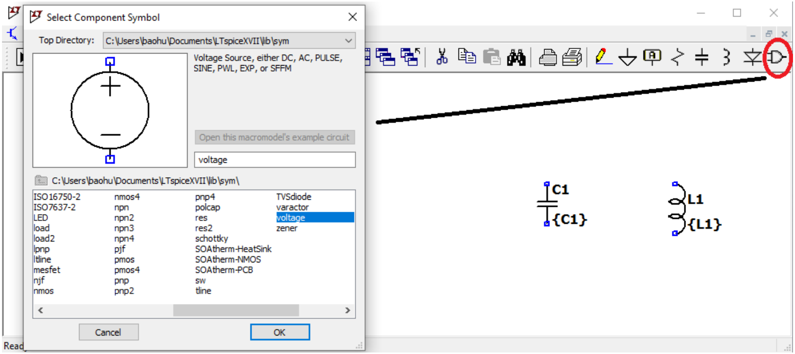
1. Point the mouse to the resistor element
2. Right-click to open the value-entry dialog
3. Enter the resistor parameter value as {R1} (note the braces) (Recommendation: enter a parameter and then declare the value, rather than entering the value directly on the element)

Example 1: Getting Started with LTspice



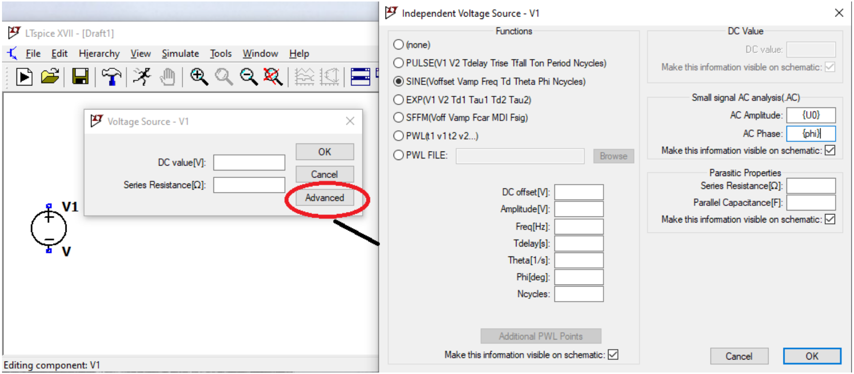
1. Continue placing and entering parameter values for L and C elements
2. To change element names and parameter values: move the mouse to the text position of the element name or value and right-click

Example 1: Getting Started with LTspice



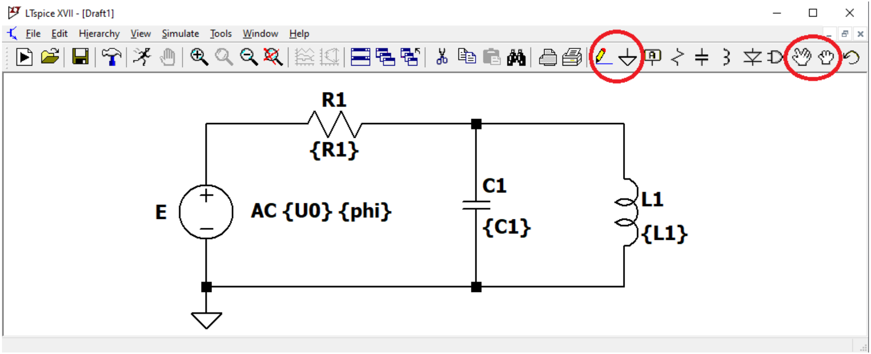
1. Click the component-library button to open LTspice's rich component library
2. Choose `voltage` to place a voltage-source element

Example 1: Getting Started with LTspice



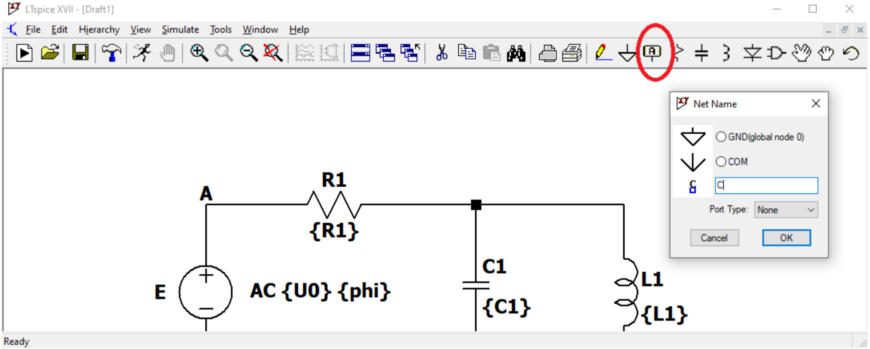
1. Right-click the voltage source → open the default DC-source dialog
2. Click `Advanced` to open the advanced dialog
3. Enter amplitude `{U0}` and phase angle `{phi}` under **AC analysis**

Example 1: Getting Started with LTspice



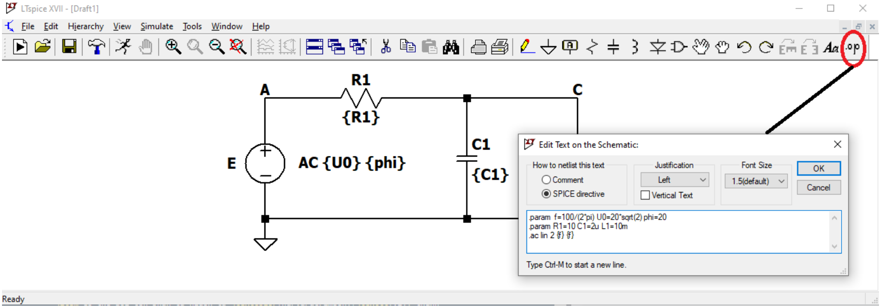
1. Use Move or Drag to arrange elements and text
   (Note the difference between Move and Drag after wiring)
2. Place the reference ground and connect wires
3. The source element name can be changed from V1 to E

Example 1: Getting Started with LTspice



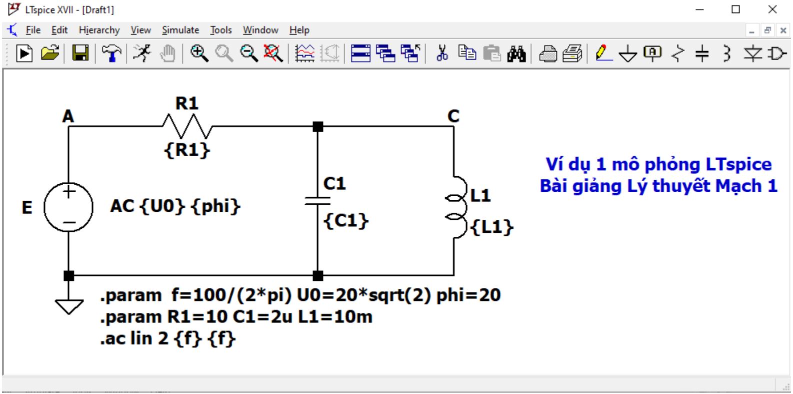
- Labels can be attached to nodes whose potentials relative to ground need to be determined
- Here, node potential A equals the source voltage E, and node potential C equals the voltage on capacitor C1 and inductor L1

Example 1: Getting Started with LTspice



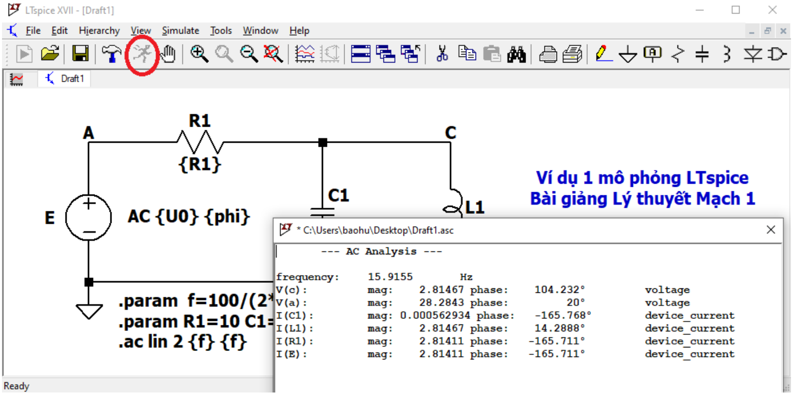
- Write parameter-value declarations `.param` and the AC simulation command `.ac` by clicking the **.op** button to open the command-entry dialog.
- Use `SPICE directive` for commands and `Comment` for comments

Example 1: Getting Started with LTspice



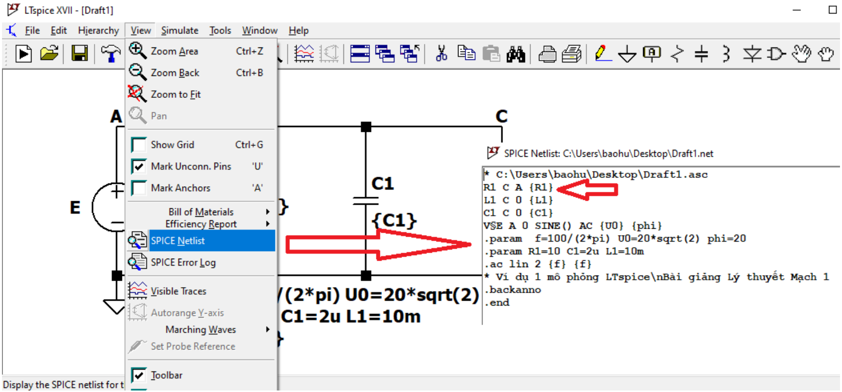
- The command `.ac lin 2 {f} {f}` solves the AC circuit with a linear frequency sweep and 2 calculation points
- In this example, the circuit is solved at one frequency, so both frequencies are set to `f`

Example 1: Getting Started with LTspice



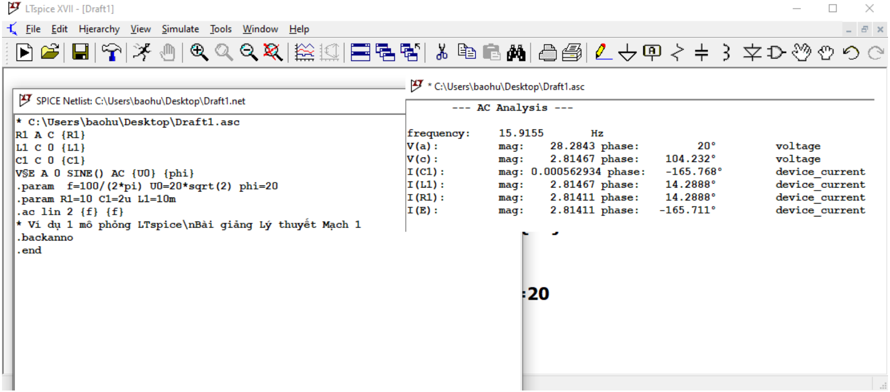
- Click **Run** to run the simulation and obtain magnitude and phase results
- Pay attention to current directions. Especially important in circuits with current-controlled sources

Example 1: Getting Started with LTspice



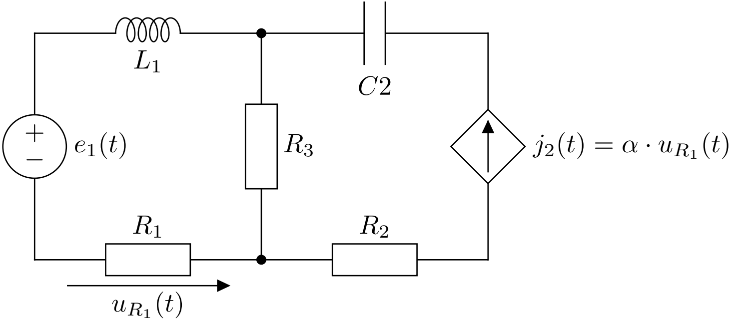
- Use **View** → **SPICE Netlist** to view the SPICE netlist
- The netlist shows that resistor R1 is connected from C to A → the direction of current $\dot{I}_{CA}$ is the positive reference direction
- The simulated current $\dot{I}_{R1}$ is opposite in phase to the positive reference current $\dot{I}_{CA}$

Example 1: Getting Started with LTspice



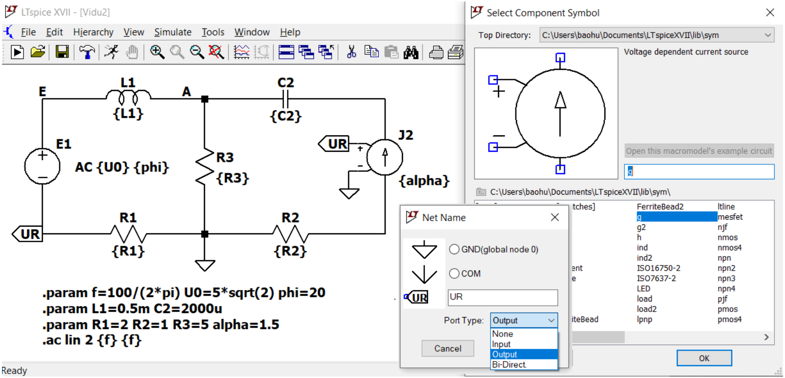
- If the positive reference direction of $\dot{I}_{R1}$ needs to be A $\rightarrow$ C, use `Move` to pull R1 off the wire, then press `Ctrl+R` to reverse R1
- Check the netlist and results again

Example 2: Voltage-Controlled Source



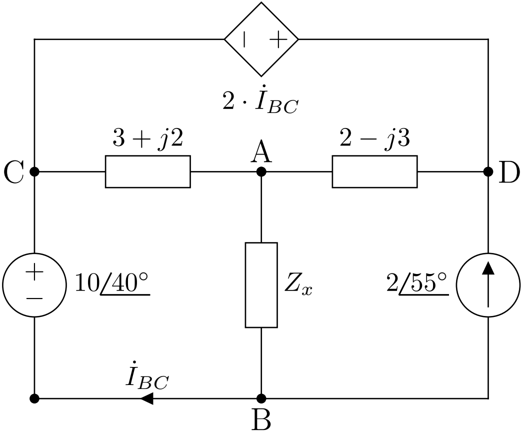
- $e_1(t) = 5\sqrt{2}\sin\left(100t + 20^\circ\right)$ V; $L_1 = 0{,}5$ mH; $C_2 = 2000\,\mu$F;
- $R_1 = 2\,\Omega$; $R_2 = 1\,\Omega$; $R_3 = 5\,\Omega$; $\alpha = 1{,}5$.

Example 2: Voltage-Controlled Source



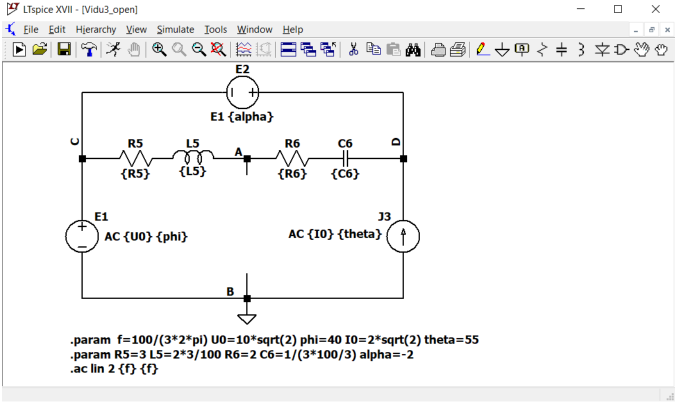
- A VCCS is element `g` (VCVS is `e`)
- Node-potential label `UR` with corresponding `output` and `input` types
- Compare with hand-determined results (note: LTspice determines amplitude, while Circuit Theory uses RMS values)

Example 3: Current-Controlled Source



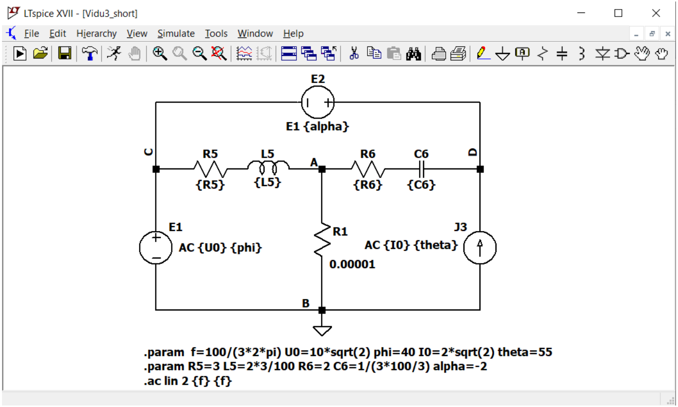
- Find Z_x such that the power delivered to this impedance is maximized

Example 3: Current-Controlled Source



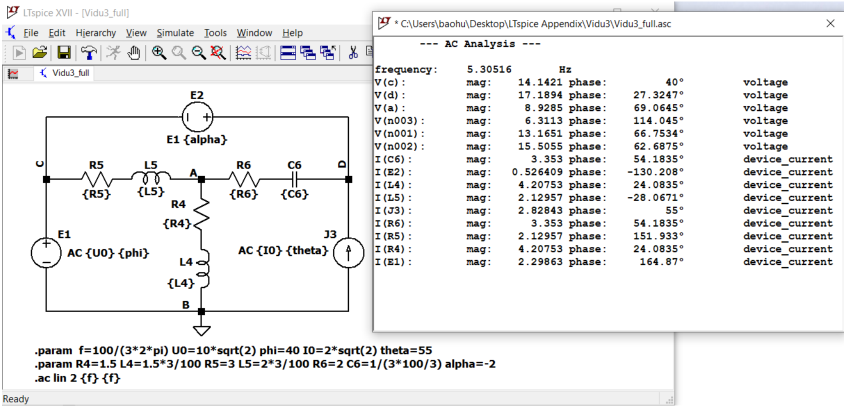
- Determine open-circuit voltage $\dot{U}_{\text{oc}} = \dot{U}_{AB}$. Assume arbitrary ω.
- A CCVS is element `h` (CCCS is `f`)
- A CC source must declare which element current it depends on (here, E1). Pay attention to sign of α w.r.s to direction of $\dot{I}_{BC}$.

Example 3: Current-Controlled Source



- Determine the short-circuit current $\dot{I}_{\mathrm{sc}} = \dot{I}_{AB}$
- LTspice determines currents through elements $\rightarrow$ use a very small resistor to approximate a wire.

Example 3: Current-Controlled Source



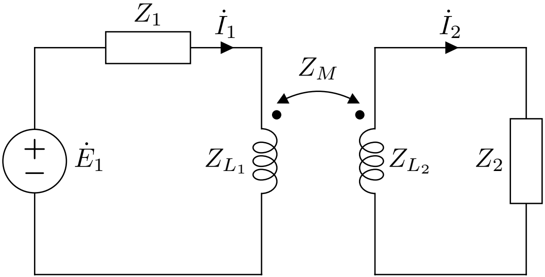
- Determine Z_{Th} and the corresponding matched load impedance
- Recheck the result with the load value just found
- Drawback: LTspice current-controlled sources are visually easy to confuse with independent sources

Example 4: Circuit with Mutual Inductance

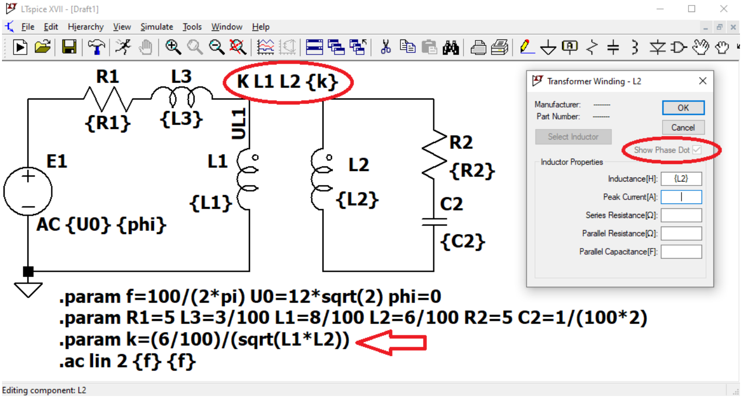
Simulate the circuit in Example 7.1

Given a transformer circuit. Determine the currents and voltages on the coils.

$\dot{E}_1 = 12\underline{/0^\circ}\,\text{V};$
$Z_1 = 5 + j3\ (\Omega);$
$Z_{L_1} = j8\ (\Omega);$
$Z_{L_2} = j6\ (\Omega);$
$Z_M = j6\ (\Omega);$
$Z_2 = 5 - j2\ (\Omega)$



Example 4: Circuit with Mutual Inductance



- Enable display of coil polarity dots
- Mutual inductance is defined between coils using the command `K L1 L2 {k}`
- Coupling coefficient $k = M/\sqrt{L_1 L_2}$